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An Oscillation Theory of Handwriting

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Abstract. Handwriting production is viewed as a constrained modulation of an underlying oscillatory process. Coupled oscillations in horizontal and vertical directions produce letter forms, and when superimposed on a rightward constant velocity horizontal sweep result in spatially separated letters. Modulation of the vertical oscillation is responsible for control of letter height. Modulation of the horizontal oscillation is responsible for control of corner shape through altering phase or amplitude. The vertical velocity zero crossing in the velocity space diagram is important from the standpoint of control. Changing the horizontal velocity value at this zero crossing controls corner shape. Changing the slope at this zero crossing controls writing slant. The corner shape and slant constraints completely determine the amplitude and phase relations between the two oscillations. This theory applies generally to a number of acceleration oscillation patterns such as sinusoidal rectangular and trapezoidal oscillations. The oscillation theory also provides an explanation for how handwriting might degenerate with speed. An implementation of the theory in the context of the spring muscle model is developed. Here sinusoidal oscillations arise from a purely mechanical source; orthogonal antagonistic spring pairs generate particular cycloids depending on the initial conditions. Modulating between cycloids can be achieved by changing the spring zero settings at the appropriate times. Frequency can be modulated either by shifting between coactivation and alternating activation of the antagonistic springs or by presuming variable spring constant springs. An acceleration and position measuring apparatus was developed for measurements of human handwriting. Measurements of human writing are consistent with the oscillation theory.

1. Introduction

Despite the biochemical complexity of the hand, there are a number of reasons why handwriting production is an attractive movement for studying human motor control. 1) There is a potentially rich set of curved trajectories that bespeaks a complex and interrelated set of motor programs. 2) Since the penpoint trajectories for connected cursive script lie on a plane and the pen itself can form part of the experimental apparatus, the measurement of handwriting trajectories is convenient and accurate. 3) Because of the speed of fast cursive script writing, handwriting trajectories are open loop and hence are a relatively pure manifestation of central programming (Denier van der Gon and Thuring, 1965; Yasuhara, 1975).

To a first approximation the hand complexity during handwriting has been factored into two functional degrees of freedom, roughly orthogonal (Denier van der Gon and Thuring, 1965; Eden, 1968; Koster and Vredenbregt, 1970; Yasuhara, 1975). In the plane of the writing surface one functional degree of freedom acts horizontally corresponding to the writing line direction, a second functional degree of freedom acts vertically corresponding roughly to the letter height direction. Usually these degrees of freedom are ascribed to the wrist and fingers, respectively, although there are exceptions and many variations. Lefthanders for example normally reverse the role of wrist and fingers. A functional degree of freedom may or may not conform to a particular joint; it could arise from a synergistic action of a number of muscles and joints. In a sense the great mechanical complexity of the hand serves not to complicate handwriting but to provide a variety of orthogonal configurations.

There are two other constraints during writing for which the additional hand degrees of freedom are necessary. The first is the requirement of holding the writing implement. The holding has to adapt to chang-

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ing joint angles; subjects find it necessary to roll the writing implement between fingers and thumb for this purpose. The second requirement is ensuring the penpoint remains on the writing surface with a particular pressure. Each constraint therefore requires some adjustment of the functional synergy depending on joint extension.

The handwriting considered here is restricted to lower case connected Palmer cursive script. We do not consider here gross positioning movements between words, between lines, for dotting *i*'s and *j*'s, and for crossing *t*'s and *x*'s. The hand degrees of freedom will be conceptually simplified to a two joint orthogonal movement; the issues of pen holding and writing pressure are not considered. The extent to which the theory presented here generalizes to other types of writing has not been investigated.

With regard to past handwriting research, the focus has been more descriptive than it has been generative. The approach to modelling a handwriting trajectory has usually been a curve fitting to measured or inferred accelerations. Mermelstein and Eden (1964) segmented writing for fitting with quarter sine waves. Denier van der Gon and Thuring (1965) assumed a rectangular form to the accelerations. McDonald (1966) fit trapezoids to the accelerations. Yasuhara (1975) assumed an exponential rise and decay time to an acceleration plateau. The end result of this process is a list of acceleration burst durations and amplitudes which when applied to the corresponding model yields synthetic writing close to the measured human handwriting. Given the great biomechanical complexity of joints and muscles, any simple exercise in exact duplication of a particular human handwritten word would seem futile. It is also unclear that curve fitting with successively more baroque models adds more enlightenment to how the human motor system accomplishes handwriting.

2. An Oscillation Theory of Handwriting

A parsimonious view of handwriting production is that handwriting arises from orthogonal oscillations horizontal and vertical in the plane of the writing surface, and that these two oscillations are superimposed on a constant velocity rightward horizontal sweep. The orthogonal oscillations are responsible for producing letter shape, while the horizontal sweep strings these letters out into a rightward moving train. The oscillations are modulated in certain ways and at specific points to produce the shapes characteristic of the English Palmer script.

It is mathematically convenient to model the oscillations by sinusoids, although the main conclusions presented here hold for other oscillation patterns such as trapezoids and rectangles. Another reason for selecting sinusoids is that later it will be seen that a spring muscle model leads to a harmonic oscillator. The equations governing the oscillations in the velocity domain can be written as:

$$\begin{aligned} \dot{x} &= a\sin(\omega_x(t-t_0)+\phi_x)+c \\ \dot{y} &= b\sin(\omega_y(t-t_0)+\phi_y), \end{aligned} \tag{1}$$

where a and b are the horizontal and vertical velocity amplitudes, ω_x and ω_y are the horizontal and vertical frequencies, ϕ_x and ϕ_y are the horizontal and vertical phases, t is the time with respect to the reference time t_0, and c is the magnitude of the horizontal sweep. These velocity equations when integrated yield a *cycloid* (Lawrence, 1972). By setting up different initial conditions a train of the basic letter shapes in handwriting is produced. The cycloids in Fig. 1 for example were produced through varying the phase. The basic shapes in handwriting demonstrated in Fig. 1 are looping (A and B in the figure), sharp (C), and rounded top corners (D and E).

By modulating the oscillation at specific times in the cycle and with specific phase and amplitude changes the oscillation train can be transformed from one basic pattern of shapes into another basic pattern. It is also clear that by modulating the vertical velocity amplitude letter height could be controlled. To understand what modulations to apply, when to apply them, and the constraints on their choice, the velocity space diagram is a useful tool (Abelson et al., 1975). The velocity equations represent an ellipse centered at $(c, 0)$. The tilt α and eccentricity e are:

$$\cot 2\alpha = \frac{b^2-a^2}{2ab\cos\phi}, \tag{2}$$

$$e = \sqrt{\frac{Q}{b^2+a^2+Q}}, \tag{3}$$

where $Q^2=(a^2+b^2)^2-4a^2b^2\sin^2\phi$, $\omega_x=\omega_y$, and $\phi=\phi_x-\phi_y$.

In Fig. 1 the velocity space diagrams corresponding to each cycloid are shown to the left; the abscissa represents $\dot{y}$, the ordinate $\dot{x}$. The ellipse tilts in Fig. 1 are the same since $a=b$, but the ellipse eccentricities vary with phase. The ellipse corresponding to $\phi=-30°$ has a clockwise rotation sense, the similar shaped ellipse with $\phi=30°$ a counterclockwise sense. Switching between counterclockwise and clockwise movement reverses the sense of top and bottom corners. The sharp top corners of letters such as *u* and *i* become the sharp bottom corners such as *m* and *n*.

2.1. Shape and Height Control

The important parameter influencing letter shape is not actually phase change but the value of the horizon-

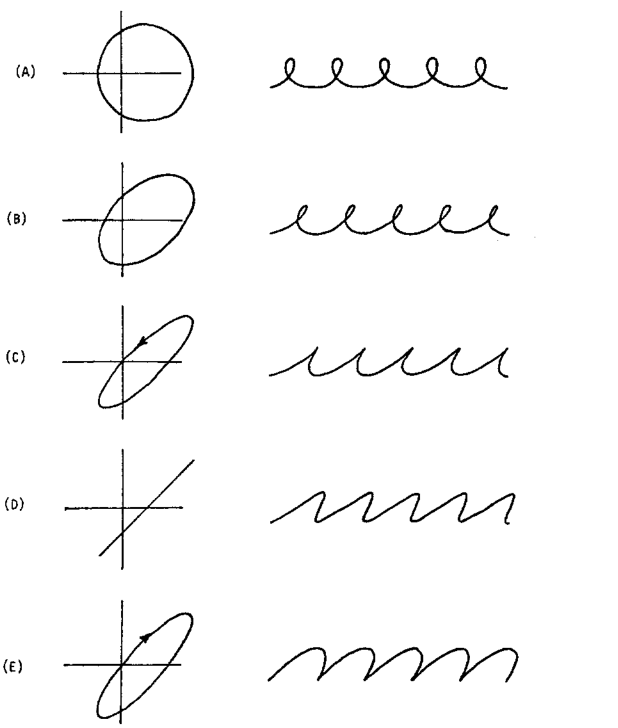


Fig. 1A–E. Phase modulation of a cycloid with parameters $a=b=2c$. The phase shifts are: **A** 90°, **B** 60°, **C** 30°, **D** 0°, and **E** −30°

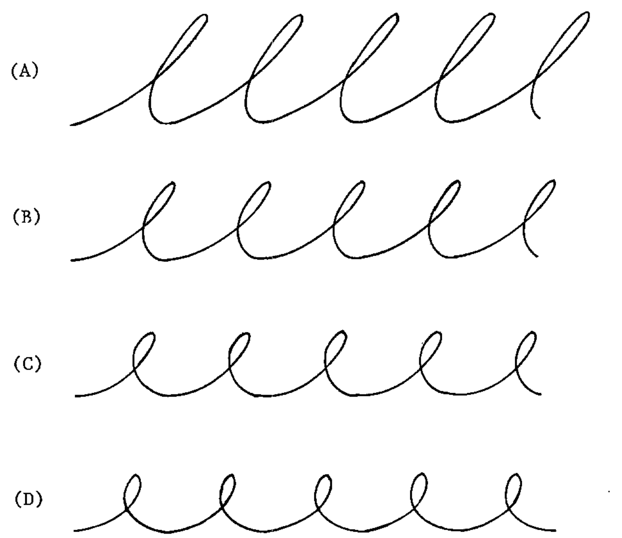


Fig. 2A–D. Cycloids generated with $a=b$, $c=20$, and with a constant vertical velocity zero crossing at $c-a\sin\phi=-14.64$, but different phase relations, indicate a similar top corner shape. Thus top corner shape is primarily controlled by the zero crossing instead of the phase difference. **A** $\phi=30°$, $a=69.28$; **B** $\phi=45°$, $a=48.99$; **C** $\phi=60°$, $a=40$; **D** $\phi=75°$, $a=35.16$

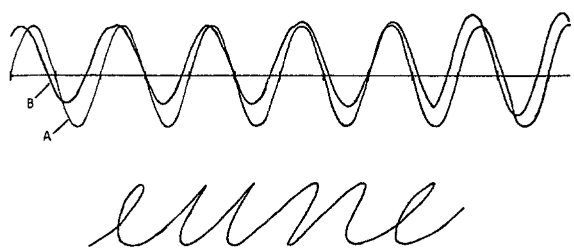


Fig. 3. Modulation of the x oscillation by adjusting the vertical velocity zero crossing produces this sequence *eune*. Above the writing, curve A is the vertical velocity, curve B the horizontal velocity

Table 1

Time	ϕ	a	$c-a\sin\phi$
0.00 s	43.66 deg	65.16 mm/s	−25.00 mm/s
0.23 s	17.03 deg	68.28 mm/s	0.00 mm/s
0.64 s	− 7.50 deg	75.64 mm/s	29.87 mm/s
0.95 s	30.94 deg	87.46 mm/s	−25.00 mm/s

tal velocity at the vertical velocity zero crossing. The vertical velocity is zero at the top corner when $\omega(t-t_0)+\phi_y=(2n+1)\pi$. The value of the horizontal velocity at this zero crossing is:

$$\dot{x}=c-a\sin\phi\,. \quad (4)$$

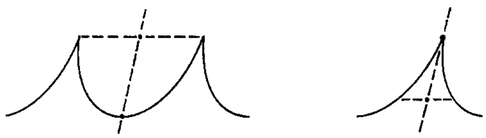
When this $\dot{x}$ is zero a sharp top corner results. The more negative this value, the fuller the top loop. The more positive this value, the rounder the top corner. To illustrate that the vertical velocity zero crossing is primarily responsible for corner shape, the sequence of e's in Fig. 2 all have the same zero crossing but differ in velocity amplitude and phase shift. It can be seen that except for letter height the top corner shape is maintained.

To transform one cycloidal pattern into another cycloidal pattern, phase and amplitude modulations of the horizontal oscillation have to be applied at appropriate times and at the appropriate levels. It is conceivable that a modulation of the vertical oscillation could assist the shape transformation such as through a phase change, but there would be added complications of keeping an even baseline and of regulating letter height. For shape modulation in which letter height does not change, therefore, it is presumed that horizontal oscillation modulation alone occurs.

As an example of shape control by altering the vertical velocity zero crossing, an e cycloid in Fig. 3 has been modulated to give the sequence *eune*. The parameters of the modulations to the horizontal oscillation are presented in Table 1. Frequency was set at 5 Hz; the vertical velocity amplitude was 28π mm/s. The units are of course arbitrary, but they have a rough physiological correspondence.

To devise the modulations, a particular value for the vertical velocity zero crossing (4), which is the last

Fig. 4. Modulation of the vertical oscillation at points of zero vertical velocity yields different letter heights by a process of amplitude modulation without phase change



(A) (B)

Fig. 5A and B. Two different methods people use in estimating writing slant. If perceived as a *u*, subjects use method **A**; if perceived as an *i*, method **B**

column in Table 1, was chosen for producing a particular corner type. Arbitrary values for a and ϕ were then chosen consistent with the desired zero crossing value. Examining the last column in Table 1, the effects of the various modulations on shape can be seen. The first modulation at 0.23 s results in a zero crossing value of 0.00 mm/s; hence a sharp top corner is obtained. The second modulation gives a positive zero crossing and a rounded top corner. Finally the original *e* cycloid is obtained through a modulation that yields a zero crossing value of -25.00 mm/s.

Letter height may be modulated by changing either the vertical acceleration amplitude or the frequency of oscillation. To preserve an even baseline, the modulation is best applied at points of zero vertical velocity, which occur at the top and bottom corners. If the point of modulation is a bottom corner, tall letters such as *l* and *h* can be produced. If the point of modulation is a top corner, lower zone letters such as *g* and *y* can be produced. An example of amplitude modulation to achieve different letter heights is illustrated by the alternating patterns of *el* in Fig. 4. Since the horizontal oscillation is unchanged and since the vertical phase does not change because the modulation was applied at a point of zero vertical velocity, corner shape has been preserved during this height modulation. The velocity space ellipses corresponding to the *e*'s and *l*'s in Fig. 4 are seen to have the same zero crossings.

2.2. Control of Writing Slant

A salient feature of Fig. 4 is the difference in writing slant between the *e*'s and *l*'s, indicating that writing slant is somehow a function of the various parameters. The horizontal and vertical oscillations act orthogonally, yet slanted writing results from their combination.

To investigate the origin of writing slant, a psychophysical experiment was performed to obtain a measure for writing slant. A group of subjects were asked to estimate writing slant in an assortment of top cusp cycloids. The subjects chose with a high degree of agreement one of two strategies in estimating slant depending on the interpretation of the cycloid. If the cycloid was interpreted as a chain of *u*'s, the slant was taken as the line from the bottom minimum point to the bisector of the line joining the two top points (Fig. 5A). If the cycloid was interpreted as a chain of *i*'s, subjects bisected the "angle" of the cusp. More exactly, the midpoints of horizontal lines joining the opposite sides of the cusp were connected (Fig. 5B). These midpoints form a straight line with slant β given by:

$$\tan\beta = \frac{b}{a\cos\phi}. \tag{5}$$

Surprisingly this slant is the same as computed in Fig. 5A; this slant is also the same as the tangent at the cusp point.

The slant measure (5) may be generalized to other writing shapes which do not have sharp top corners such as *e*'s, and corresponds to the slope of the velocity space ellipse at the vertical velocity zero crossing. The importance of this measure is that it shows writing slant is an artifact of an orthogonal oscillation system. The writing slant changes as the horizontal velocity a, the vertical velocity b, and the phase difference ϕ vary. Reexamining the difference in slant between tall and short letters in Fig. 4, Eq. (5) yields an *e* slant of 61.8° and an *l* slant of 79.1°. The change in slant arises because the vertical velocity amplitude b changed.

Assuming that stylistic constraints require a constant writing slant, one must exercise care in going between tall and short letters to maintain slant. Since in letter height modulation the vertical velocity amplitude b is changed, it is required that the horizontal velocity amplitude a and phase difference ϕ also change to maintain a constant slant β. The constraints on letter shape and letter slant can be expressed by constants k_1 and k_2 respectively as follows

$$\begin{aligned} k_1 &= c - a\sin\phi \\ k_2 &= \frac{b}{a\cos\phi}. \end{aligned} \tag{6}$$

From these two relations the letter height constraint implies that

$$b = k_2(c - k_1)\cot\phi. \tag{7}$$

Table 2

Plot	a	ϕ	b
A	29.28	75	10.72
B	32.66	60	23.09
C	40.00	45	40.00
D	56.57	30	69.28
E	109.28	15	149.28

If a particular letter height is stipulated (namely $2b/\omega$), then the phase difference ϕ and the horizontal velocity amplitude a are determined. A plot of different e cycloids satisfying (7) but with different phases and velocity amplitudes is presented in Fig. 6; Table 2 lists the relevant parameters. The perceived slant and shape seem to be constant among the several plots.

An alternating pattern of e's and l's that preserves both slant and shape under amplitude modulation alone is illustrated in Fig. 7. Examining the velocity space ellipses, both the zero crossings and the slopes at the zero crossings are the same for the e and l ellipses.

The slant Eq. (5) is independent of frequency, and it would seem that if height control were obtained with frequency modulation alone the control of slant would be simplified. This idea works fine for the vertical oscillatio, but the shape constraint forces the horizontal oscillation to modulate both frequency and acceleration amplitude. It will be seen later that experimental measurements on humans show a combined amplitude and frequency modulation to control letter height, with frequency modulation the predominant influence.

The previous discussion reveals that the vertical velocity zero crossing is of paramount importance in the control of handwriting (Fig. 8). To control writing shape the value of the horizontal velocity at this zero crossing is altered. To control writing slant the slope of the velocity space diagram at the zero crossing is altered. Writing height controlled either by frequency modulation, which cannot be distinguished in the velocity space diagram, or by amplitude modulation, which affects the vertical elongation of the velocity space diagram.

2.3. *Alternative Oscillation Patterns*

The forcing function need not be sinusoidal for the above conclusions to hold. For example, a rectangular acceleration pattern (no rise time) yields triangular velocity profiles and a parallelogram in velocity space (Fig. 9A). The writing produced by this oscillation pattern is a quite acceptable chain of u's. Using once again the slope of the velocity space ellipse at the vertical velocity zero crossing as a measure of writing slant, the slant angle is given by $\tan\beta = b/a$. Psychophysical experiments for writing slant have not

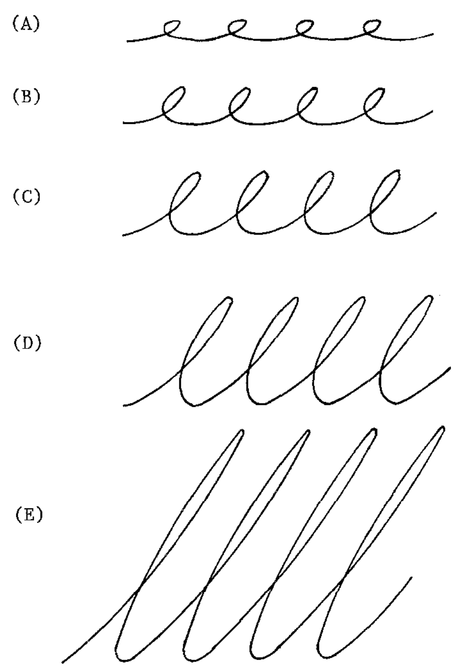


Fig. 6A–E. A sequence of e cycloids with constant slant measure β and constant vertical velocity zero crossing but different amplitude and phase values. The perceived slant and shape appear the same; the parameters for these cycloids appear in the text

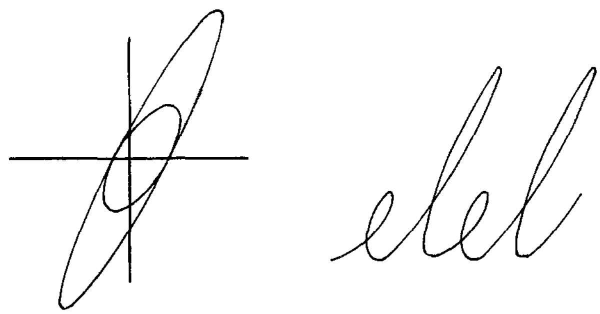


Fig. 7. Letter height modulation under constraints of constant slant and vertical velocity zero crossing is achieved with amplitude modulation. The velocity space ellipses on the left show that the vertical velocity zero crossing value and slope are the same for both e and l ellipses

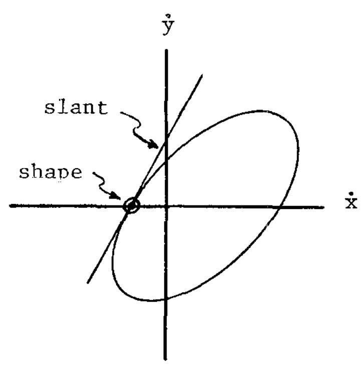


Fig. 8. Manipulation of the velocity space diagram at the vertical velocity zero crossing leads to *(1)* control of shape by altering the value of the zero crossing, and *(2)* control of slant by altering the slope at the zero crossing

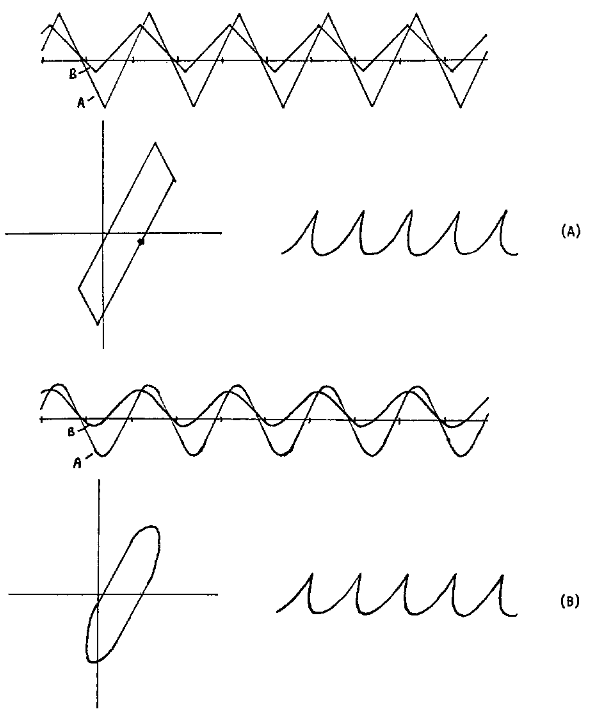


Fig. 9. A Triangular velocity profiles resulting from a rectangular acceleration pattern; B the velocity profiles are rounded with a trapezoidal acceleration pattern

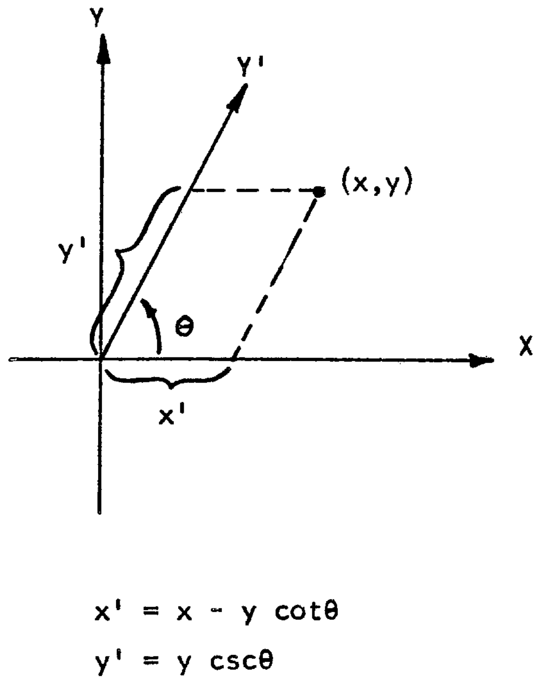


Fig. 10. Shear transformation for a nonorthogonal joint angle θ. The equations governing this transformation are: $x' = x - y \cot\theta$, $y' = y \csc\theta$

been performed with synthetic writing produced by rectangular acceleration patterns to ascertain the validity of this measure. If this measure is accurate, the independence of slant on phase is noteworthy. The advantage of phase independent writing slant is a considerable simplification of control.

Trapezoidal peaks have also been proposed as models of the acceleration patterns in handwriting (McDonald, 1966). The slopes to the plateaus presumably model a linear rise of the actuation. The effect of the linear rise time on the velocity space diagram is to round the corners of the rectangular pattern's velocity space parallelogram (Fig. 9B). If the rounding does not occur at the vertical velocity zero crossing the writing slant is again given by $\tan\beta = b/a$. If the rounding occurs through the zero crossing the writing slant becomes phase dependent as for the sinusoidal case.

2.4. *Nonorthogonal Writing Axes*

Other explanations for writing slant have been put forth. Mermelstein (1964) proposed that a nonorthogonal axes disposition could explain writing slant. The slant of the handwriting would follow the direction of the slanted vertical axis. From the previous discussion it is clear that coupled oscillations cause as a side effect slanted writing, and that therefore a nonorthogonal joint disposition cannot alone account for writing slant. Any slant from nonorthogonal axes must be added to the slant caused by coupled oscillations.

There is a question as to how one can distinguish these two contributions. If θ is the angle the vertical axis makes with the horizontal axis, and if cycloidal velocity equations (1) act along the nonorthogonal axes, then the orthogonally measured velocities are (see Fig. 10):

$$\begin{aligned}\dot{x} &= a\sin(\omega t + \phi) + c + b\cos\theta\sin\omega t\\ \dot{y} &= b\sin\theta\sin\omega t\,.\end{aligned} \tag{8}$$

The velocity space diagram corresponding to (8) is also an ellipse but with altered tilt and eccentricity. It is impossible to factor the product $b\cos\theta$ by manipulations on the measured velocities. That is to say, there are infinitely many pairs (b, θ) that yield exactly the same writing. This multiplicity may actually pose an advantage for situations where the joint disposition changes within a word; for example, in going between extremes of a joint before repositioning. The writing however can be kept the same merely by adjusting b. For the present the extent of any contribution to slant from nonorthogonal axes remains an open question.

Another explanation for writing slant was put forth by Pellionisz and Llinas (1979), who proposed that writing slant arose from computing delays in the cerebellum. According to their theory an idealized up-down movement gets transformed into a slanted line because the trajectory computation lags the actual trajectory.

3. Handwriting under a Spring Model

The oscillation theory indicates how handwriting can be produced. The next step is to propose how the theory could be implemented specifically by the human motor system. Particular issues that have to be resolved are the agent responsible for the oscillation, the nature of the control variables, and the influence the control variables have on bringing about the various modulations.

A recent theory of motor control likens muscle to a spring system with variable zero setting (Feldman, 1974a, b). This spring muscle model has a particular affinity to the oscillation theory, and provides a useful framework for considering one way in which the human motor system might implement this theory. In this section the issues of parameter control raised in the previous paragraph are developed in the context of the spring muscle model.

The spring muscle model is a simplification of the length-tension curves of muscle. In Fig. 11 length-tension curves under isometric contraction at several firing rates for the cat soleus muscle are shown (Rack and Westbury, 1969). The spring muscle model is derived by assuming muscle operates at the linear short length regions. The length-tension curves in this region are also assumed parallel, although Fig. 11 shows a slightly increasing slope with firing rate. The simplified length-tension curves of agonist and antagonist muscles would intersect and overlap as in Fig. 12; different length-tension curves are selected by adjusting the muscle firing rate.

Feldman (1974a, b) has proposed that movements are executed by selecting a pair of agonist-antagonist length-tension curves that intersect at the desired position. This process is illustrated in Fig. 12. Suppose the system is currently at length L_0 under innervation rates g_1 for the agonist and n_1 for the antagonist. If the innervation rate of the agonist is changed to g_3, a different agonist length-tension curve is selected and the equilibrium length shifts to L_1. Assuming no delay in tension development and ignoring velocity effects, the arrow in the figure indicates the tension course. There is an isometric buildup of tension from P_0 to P_2 followed by a decay to P_1, where the tension in agonist balances the tension in antagonist.

The curves in Fig. 12 lead to a model of muscle as a spring with variable zero setting. The slope K of the curves represents the spring constant, and the variable zero setting L_z corresponds to the selection of firing rate. The force exerted by a muscle is thus $K(L-L_z)$. In

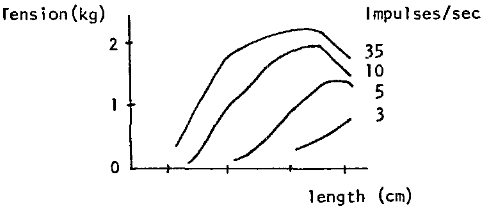


Fig. 11. Length-tension curves from the cat soleus muscle, from Rack and Westbury (1969)

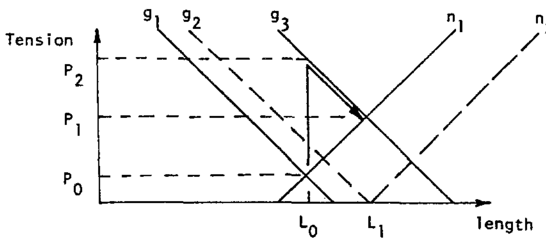


Fig. 12. The equilibrium point of the intersecting length-tension curves of agonist (g labels) and antagonist (n labels) shifts from L_0 to L_1 when the firing rate of the agonist is raised from g_1 to g_3 and the antagonist rate remains at n_1

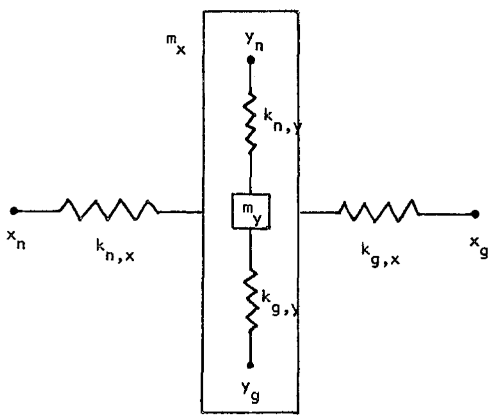


Fig. 13. Two orthogonal pairs of springs serve as a model for handwriting. The y springs act on the pen mass m_y, the x springs act on the mass m_x which is the sum of the pen mass and the y spring platform

the remaining discussion viscous and passive elastic components of muscle are neglected in order to develop the main points.

If handwriting is executed by two orthogonal joints, we can model the system by the action of two orthogonal opposing pairs of springs (Fig. 13). The y springs act on the pen mass m_y; the x springs act on the pen mass and on the y spring masses themselves, indicated by attachments to a platform of total mass m_x mounted with the pen mass and the y springs. Neglecting viscous friction, the equation of motion of the pen mass in the y direction is:

$$m_y\ddot{y}=k_{g,y}(y_g-y)-k_{n,y}(y-y_n)\,. \tag{9}$$

Absorbing the mass m_y into spring constants and solving this equation by Laplace transforms:

$$y(t)=\frac{-b}{\omega_y}\cos(\omega_y(t-t_0)+\phi_y)+\frac{k_{g,y}y_g+k_{n,y}y_n}{\omega_y^2}, \tag{10}$$

where $\omega_y^2=k_{g,y}+k_{n,y}$ and

$$b\sin\phi_y=\dot{y}(t_0),$$

$$b\cos\phi_y=\omega_y\left(\frac{k_{g,y}y_g+k_{n,y}y_n}{\omega_y^2}-y(t_0)\right).$$

According to the oscillation theory the horizontal movement has a constant velocity sweep superimposed on an oscillation. The justification as discussed later comes from the observation that the velocity space diagrams stay centered about the same point through all the different modulations of velocity for different letter shapes. The constant velocity sweep is imposed on any additional change to the x spring zero settings. Setting $x_g(t)=x_g+c(t-t_0)$ and $x_n(t)=x_n+c(t-t_0)$ and solving the equations corresponding to (9),

$$x(t)=\frac{-a}{\omega_x}\cos(\omega_x(t-t_0)+\phi_x)+\frac{k_{g,x}x_g+k_{n,x}x_n}{\omega_x^2}+c(t-t_0), \tag{11}$$

where $\omega_x^2=k_{g,x}+k_{n,x}$ and

$$a\sin\phi_x=\dot{x}(t_0)-c,$$

$$a\cos\phi_x=\omega_x\left(\frac{k_{g,x}x_g+k_{n,x}x_n}{\omega_x^2}-x(t_0)\right).$$

When differentiated, Eqs. (10) and (11) are the same as the velocity sinusoids of Eqs. (1) and (2). A sinusoidal oscillation arises from the spring muscle model as the simplest method of operation. Once an appropriate set of initial conditions is set up, the oscillation propagates indefinitely with the zero settings unchanged. With the orthogonal spring pairs, a cycloid ensues in the manner of Fig. 1. The agent for the oscillation under the spring model is a purely mechanical one, assuming of course no dissipation from viscous elements. If viscous elements were to be included, an active involvement of the control centers to maintain the mechanical oscillation would be required.

3.1. Modulating Letter Shape and Height with Springs

The previous section indicated that controlling the vertical velocity zero crossings and controlling the vertical velocity amplitudes of a cycloid result in modulation of shape and modulation of height. The vertical velocity zero crossing was shown to be a function of the horizontal velocity amplitude and phase. In the spring model the amplitudes and phases are controlled by adjusting the zero settings of the springs. A difficulty under this spring model is that phase and amplitude cannot be controlled independently. In order to achieve a particular zero crossing, for example, one has to search for the particular zero setting applied at a particular time that yields the desired product $a\sin\phi$. In order to modulate letter height the modulation is best applied at the bottom corner at zero vertical velocity; the resulting vertical amplitude changes without phase change.

It is a potential difficulty with the spring model if a physical situation exists with $\omega_x \neq \omega_y$ and there is no way to make these frequencies equal. For example, the mass of the writing implement, the size of the limbs involved, and the frictional contact with paper are subject to change and influence the frequencies. This difficulty implies a need to be able to adjust frequencies and hence to vary the spring constants. A variable spring constant model is considered in Sect. 3.3. For the present discussion we assume fixed spring constants with $k=k_{g,x}=k_{n,x}=k_{g,y}=k_{n,x}$ and $\omega=\omega_x=\omega_y$.

A final issue before considering corner shape and letter height modulation is that a change in zero setting can be applied to either the agonist or antagonist spring or to both at the same time. As far as the mathematics is concerned these situations are all equivalent. Examining for example the y Eq. (10), the influence of the zero settings is given by $(y_g+y_n)/2$ after taking into account the simplifications of the previous paragraph. Thus for a given axis there is only one functional control parameter, corresponding to the combined change in zero settings of both springs. There do arise situations where a zero setting change can be applied only to one spring in order to avoid the other spring from pushing as well as pulling. Such situations will not arise in the following examples, and for simplicity we will assume the change in zero setting Δy or Δx is applied to the agonist spring.

To illustrate that the vertical velocity zero crossing can be manipulated by changing the horizontal agonist zero setting, the increments Δx to x_g required to produce the *eune* of Fig. 3 are presented. The *e* cycloid is generated with initial conditions in Table 3. The three increments Δx that bring about phase and amplitude changes in Table 1 corresponding to the *u*, the *n*, and the *e* are 1.967, 2.0, and 3.493 mm respectively. In fact the writing in Fig. 3 was generated by the spring model.

We consider next how to produce the amplitude modulation for letter height that yields the alternating *e*'s and *l*'s of Fig. 8. There are three constraints on the letter height modulation: (1) the achievement of a particular letter height, in this case a ratio of $2\sqrt{2}$

Table 3

Parameter	Value	Parameter	Value
$x(t_0)$	$-$ 1.5 mm	$y(t_0)$	-2.8 mm
$\dot{x}(t_0)$	65.0 mm/s	$\dot{y}(t_0)$	0.0 mm/s
x_g	4.0 mm	y_g	5.0 mm
x_n	$-$ 4.0 mm	y_n	-5.0 mm
c	20.0 mm/s	ω	5.0 c/s

Table 4

Parameter	$t=0.0$ s	$t=0.2$ s
Δy	0.0 mm	10.24 mm
Δx	0.0 mm	5.37 mm
b	28π mm/s	248.8 mm/s
a	65.16 mm/s	140.7 mm/s
ϕ	0.762 rad	0.326 rad

(close to the human average), (2) a constant writing slant, and (3) a constant corner shape. Fortunately there are an equal number of control variables, namely the zero setting changes Δx and Δy, and the time t_1 of the Δx application. The interletter separation is fixed by these choices of parameters and is not under independent control.

As mentioned earlier the y modulations are best applied at points of zero vertical velocity, because the phase change is then always zero and the maintenance of an even baseline is simplified. Under acceleration amplitude modulation for letter height the writing size is directly proportional to the acceleration amplitude. The time of horizontal modulation is found to coincide with the point of vertical modulation. The increments Δy and Δx to the vertical and horizontal agonist springs to produce the first l of Fig. 8 is presented in Table 4, along with the amplitudes and phases of the corresponding cycloids. The slant of the resultant l is the same as the slant of the e and the vertical velocity zero crossing has the same value for the l and the e.

Ordinarily when modulating for height in upper zone letters such as l or b no special corner shaping need be done by the horizontal springs. For lower zone letter such as y and j on the other hand a considerable degree of corner shaping is required. Obtaining a lower zone loop as in the letter y (Fig. 14) involves a transition from counterclockwise to clockwise movement, then back from clockwise to counterclockwise movement. A difficulty resulting from these transitions is creating enough horizontal separation between the y and the next letter e, and so the bottom of the y must curl around to make the upstroke more propitious. To achieve these transitions and spacing requires 4 horizontal zero setting increments Δx in a span of 150 ms, bringing about large changes in phase and very large horizontal velocity amplitudes (Table 5).

If one can speak about the relative difficulty in producing various letters, the number of transitions between counterclockwise and clockwise movements would indicate the level of difficulty. Associated with such a transition are usually large phase and amplitude changes, and one might construct a complexity measure based on these changes. By any such measure lower zone loops as in y would have to be considered

Table 5

Time	Δx	$\Delta\phi$	a'
0.43 s	2.0 mm	24.5 deg	67.5 mm/s
0.74 s	-11.0 mm	67.0 deg	164.6 mm/s
0.80 s	11.0 mm	-24.6 deg	309.6 mm/s
0.85 s	-12.0 mm	-47.7 deg	312.7 mm/s
0.89 s	11.0 mm	-21.7 deg	74.5 mm/s

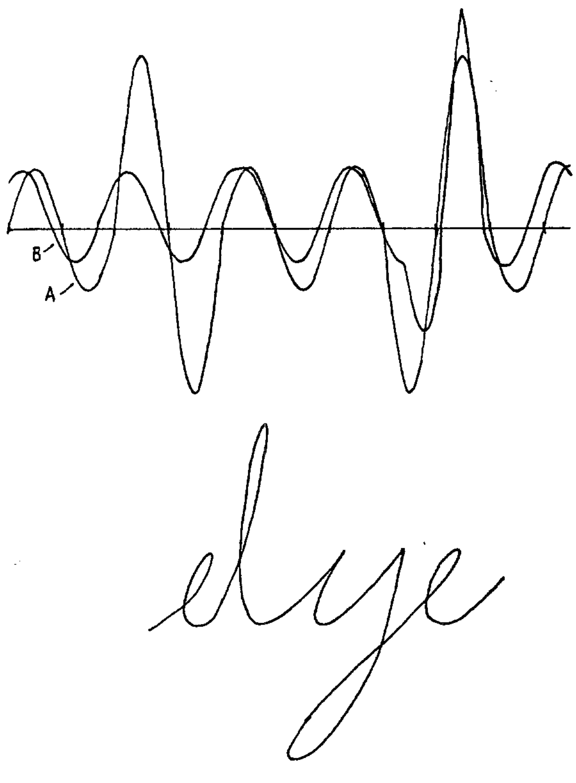


Fig. 14. Amplitude modulation is used to produce short and tall letters in this sequence *elye*. Note the change in slant between short and tall letters

difficult. Another difficult form is the letter h (Fig. 15), which requires two counterclockwise/clockwise transitions. The first transition goes into making the bottom cusp coming from the top loop, accomplished with 2 closely spaced modulations resulting in fairly large phase changes (Table 6). The second transition restores the sequence of e's, but at the cost of a very large phase change.

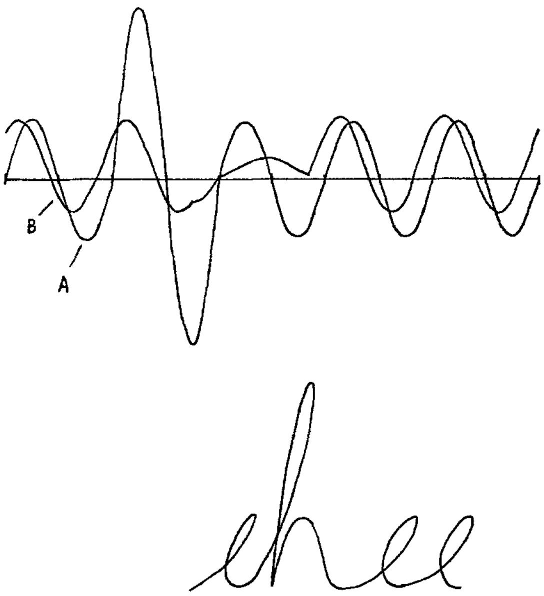


Fig. 15. The letter h presents problems in production because of the need for several large phase changes

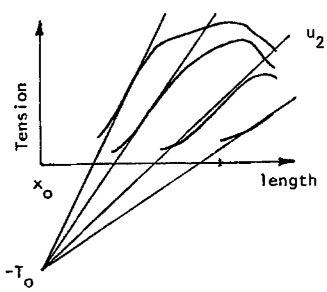


Fig. 16. The length-tension curves of Rack and Westbury (1969) when extended meet at a point. The dependence of the slope $k_{n,x}$ on the adjustable zero setting x_n can be characterized by $T_{n,x}/(x_n - x_{n,0})$

In the next section some evidence will be presented that fast human handwriters achieve speed by simplifying the script to eliminate all clockwise movements. If one may draw an analogy between spring writing and human writing, the reason might be that counterclockwise/clockwise transitions are too difficult to make fast.

3.2. *Variable Spring Constant Model*

The previous discussions took place in the context of a spring muscle model with fixed stiffness. There is reason to presume that the human motor system can control not only muscle tension but also muscle stiffness. There are several reasons why stiffness regulation would be desirable from the standpoint of the spring handwriting model. 1) There is a need to equate horizontal and vertical oscillation frequencies under different physical conditions, as mentioned earlier. 2) Stiffness regulation would provide another control

Table 6

Time	Δx	$\Delta\phi$	a'
0.35 s	−5.0 mm	65.2 deg	50.7 mm/s
0.40 s	−2.0 mm	55.2 deg	20.3 mm/s
0.57 s	5.0 mm	−113.0 deg	69.3 mm/s

Table 7

Parameter	Value	Parameter	Value
$x_{g,0}$	20.0 mm	$y_{g,0}$	20.0 mm
$x_{n,0}$	−20.0 mm	$y_{n,0}$	−20.0 mm
x_g	17.5 mm	y_g	17.5 mm
x_n	−17.5 mm	y_n	−17.5 mm
$T_{g,x}$	1233.7 mm/s^2	$T_{g,y}$	1233.7 mm/s^2
$T_{n,x}$	1233.7 mm/s^2	$T_{n,y}$	1233.7 mm/s^2
$x(t_0)$	−0.51 mm	$y(t_0)$	−1.0 mm
$\dot{x}(t_0)$	23.2 mm/s	$\dot{y}(t_0)$	0.0 mm/s
c	7.15 mm/s		

variable and would allow more flexibility in modulation. 3) If letter height could be modulated by frequency instead of by amplitude, then the slant Eq. (4) implies that the slant would not change because slant is not a function of frequency. The next section provides evidence that people use in part a strategy of frequency modulation for letter height.

We take as an analytic model of stiffness variation with slant an extrapolation from the length-tension curves of Rack and Westbury (1969). If the linear portions of these curves are extended they seem to meet at a point (Fig. 16). For the horizontal antagonist spring, for example, the dependence of the spring constant $k_{n,x}$ of a variable stiffness spring corresponding to this ray of straight lines is:

$$k_{n,x} = \frac{T_{n,x}}{x_n - x_{n,0}}, \tag{12}$$

where $x_{n,0}$ is the length at which the rays intersect and $-T_{n,x}$ is the tension at $x_{n,0}$. It should be emphasized that the exact dependence of stiffness on zero setting is not important for analyzing handwriting under a variable spring constant model, and the relation (12) is chosen merely because it is convenient and because the literature does not contain an alternative expression.

We reexamine the issue of control of letter height, slant, and shape raised in Sect. 3.2. A set of initial conditions appropriate for this model are presented in Table 7. With these parameters a train of e's is produced with the same shape as produced by the fixed spring model and the parameters of Table 3. When modulating for letter height human subjects typically decrease the frequency by a factor of $\sqrt{2}$ and double the vertical velocity amplitude. The increments Δy_g

and Δy_n to the agonist and antagonist y springs can be chosen to satisfy both conditions (Table 8). When modulating the horizontal springs to preserve slant and shape, the horizontal frequency must also be made to match the vertical frequency. The horizontal modulation is forced by the mathematics to occur at the same time as the vertical modulation, and the increments Δx_g and Δx_n to the agonist and antagonist x springs are given in Table 8. The resultant writing appears in Fig. 17. The shape and slant constraints have been met. Table 8 shows that the x and y frequencies have been decreased by a factor of $\sqrt{2}$, and the vertical velocity amplitude has been doubled.

A frequency decrease by $\sqrt{2}$ and a doubled vertical velocity amplitude implies that the vertical velocity amplitude increase is due equally to a frequency modulation and an acceleration amplitude modulation. That is to say, a constant acceleration amplitude is not maintained through a height modulation. Nevertheless the idealization of achieving height modulation without slant change by frequency modulation would have been obtained if the horizontal velocity amplitude could have been doubled also and the phase difference kept constant. Unfortunately the shape constraint prevents this easy solution for slant control, since Eq. (6) requires the phase difference to change and the horizontal velocity amplitude to assume some value other than double the original amplitude. Nevertheless there has been a net benefit in reducing the "difficulty" of letter height modulation because both the horizontal and vertical velocity amplitude changes are half that in the fixed spring model, and the required phase change is smaller.

3.3. Coactivation Versus Alternate Activation

So far the spring model has been presumed to work by having both springs exert force at the same time. It is conceivable to have only one spring on at a time. In the simplest mode of operation the zero settings x_g and x_n or y_g and y_n would be equal, and whenever the position passes through the zero setting the agonist spring is switched off and the antagonist spring on. The equations of motion are exactly the same as before, except that the frequency is smaller by a factor of $\sqrt{2}$. The control of both systems therefore is equivalent. It is possible to have a mode of operation where the antagonist spring comes on before the point of maximum velocity, but analysis of such controls shows that maintenance of an even baseline is rather difficult because of the complexity of the equations.

It is possible to use a scheme of switching between simultaneous and alternate agonist/antagonist spring activation to achieve a limited frequency modulation under a fixed spring constant model. A frequency

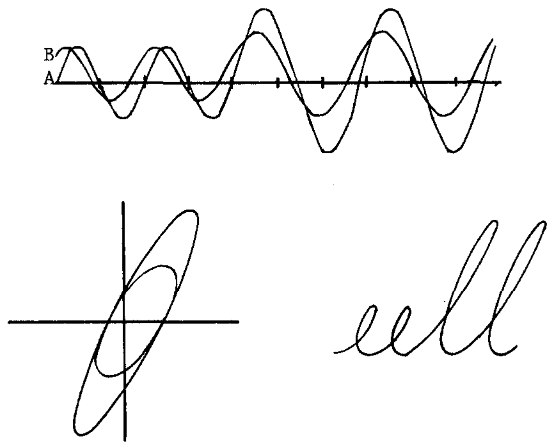


Fig. 17. Letter height control by combined frequency and amplitude modulation in a variable spring model is illustrated by the synthetic writing in the bottom right diagram. *Top diagram:* vertical (*A*) and horizontal (*B*) velocity traces, showing frequency and amplitude modulation. *Bottom left:* velocity space diagram shows slant and shape constraints have been met

Table 8

Parameter	0.0 s	0.4 s
Δy_g	0.0 mm	$-$ 2.08 mm
Δy_n	0.0 mm	3.00 mm
$k_{g,y}$	493.5/s	269.3/s
$k_{n,y}$	493.5/s	224.2/s
ω_y	10π/s	$5\pi\sqrt{2}$/s
b	10π mm/s	20π mm/s
ϕ	$\pi/4$ rad	0.464 rad
Δx_g	0.0 mm	$-$ 2.28 mm
Δx_n	0.0 mm	2.75 mm
$k_{g,x}$	439.5/s	258.2/s
$k_{n,x}$	439.5/s	235.2/s
ω_x	10π/s	$5\pi\sqrt{2}$/s
a	22.7 mm/s	35.9 mm/s

decrease by $\sqrt{2}$ would by itself result in a doubled letter height. Any additional height modulation would require changing the vertical acceleration amplitude.

4. Experimental Measurements with Humans

In this section an apparatus designed for accurate measurement of position, velocity, and acceleration during human handwriting is described. Measurements of human handwriting with this apparatus are then compared against the oscillation model.

4.1. Acceleration Measurement

Measurements of acceleration during handwriting are obtained with accelerometers mounted on a 6 degree of freedom X-Y sliding rail system (Fig. 18). A linear

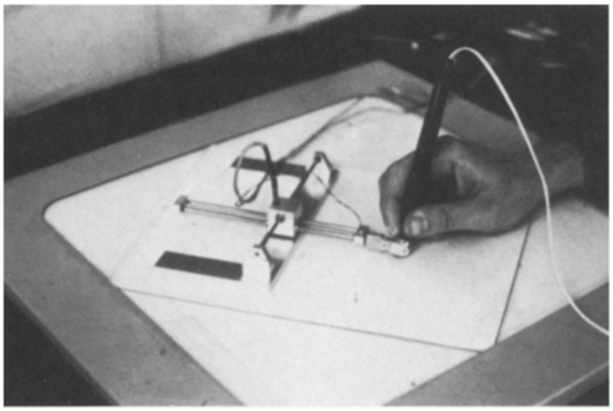


Fig. 18. A six degree of freedom $x-y$ rail system for measurement of acceleration and position

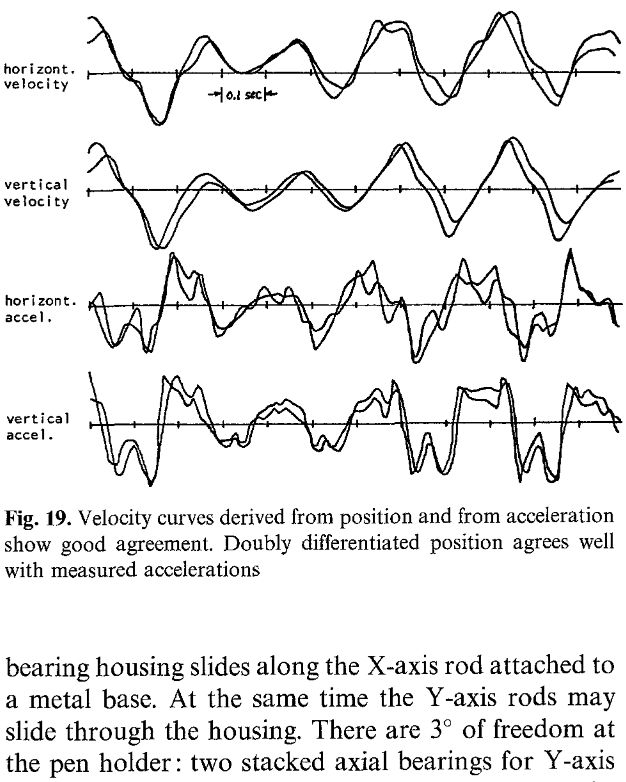


Fig. 19. Velocity curves derived from position and from acceleration show good agreement. Doubly differentiated position agrees well with measured accelerations

bearing housing slides along the X-axis rod attached to a metal base. At the same time the Y-axis rods may slide through the housing. There are 3° of freedom at the pen holder: two stacked axial bearings for Y-axis rotation, a hinge joint for rotation with respect to the end of the Y-axis rods, and a bronze bushing for rotation of the pen inside the holder. Play in the apparatus is negligible.

The whole apparatus allows nearly frictionless movement. Subjects reported no constraints on their writing movement and no difficulties in grasping the pen near t he holder. Tests showed that the inertia of the apparatus did not affect the writing act. The finger and wrist muscles are evidently overpowered for the handwriting movement.

The acceleration is sampled at a rate of 250 coordinate pairs per second at a resolution of better than 0.01 g's. A simple RC filter to the accelerometers with cutoff frequency 25 Hz removes high frequency paper noise. It is followed by a triangular digital filter with Nyquist frequency 25 Hz.

This apparatus has some advantage over previous acceleration measurement apparatuses. Designs with accelerometers or force sensors mounted in the pen (Crane and Savoie, 1977; Herbst and Liu, 1977) are susceptible to orientation problems. Subjects roll the pen in their fingers as they write and also change the pen inclination; the sensitive axes of the accelerometers or force sensors change unpredictably and the data loses its meaning. Constraints imposed on subjects to obviate orientation problems interfere with the writing act. A second class of apparatuses derive acceleration from position; these include writing tablets, electrolytic water tanks (McDonald, 1966; Yasuhara, 1975), teledelthos spark systems (Denier van der Gon and Thuring, 1965), and a sliding rail system with potentiometers (Koster and Vredenbregt, 1971). The time and spatial resolution of these apparatuses are not currently sufficient for accurate acceleration derivation; there is no choice but to measure acceleration directly. The sliding rail system employed in this thesis obviates the orientation problems with pen mounted systems, and provides an accurate and detailed record of the acceleration of the pen tip.

4.2. *Position Measurements*

Position measurements are obtained with a Summagraphics ID Data Tablet/Digitizer. The writing tablet pen fits into the holder of the accelerometer apparatus. Velocity is estimated by differentiating position. Velocity estimated by integrating acceleration shows close agreement to the position derived velocities (Fig. 19). Also present in the figure is the matching of measured acceleration to doubly differentiated position, again showing a good agreement. The sampling rate for position is 94 coordinate pairs per second at a spatial resolution of 0.1 mm. A narrow triangular filter was applied to the position data to yield a smoothed velocity curve. A complete recording of a handwriting movements is illustrated in Fig. 20.

The sharp negative acceleration peaks in Fig. 20 have been attributed to static friction at the tops of strokes where the velocity is near zero (McDonald, 1966; Yasuhara, 1975). My own experiments on the other hand indicate that the separate peaks are at least not completely a friction artifact. Subjects were asked to make an up-down finger movement on plexiglass. With this arrangement there is negligible friction between pen and surface and between hand and surface since the fingers only are moving off the surface. Sharp peaks are evident in the resultant acceleration recording (Fig. 21).

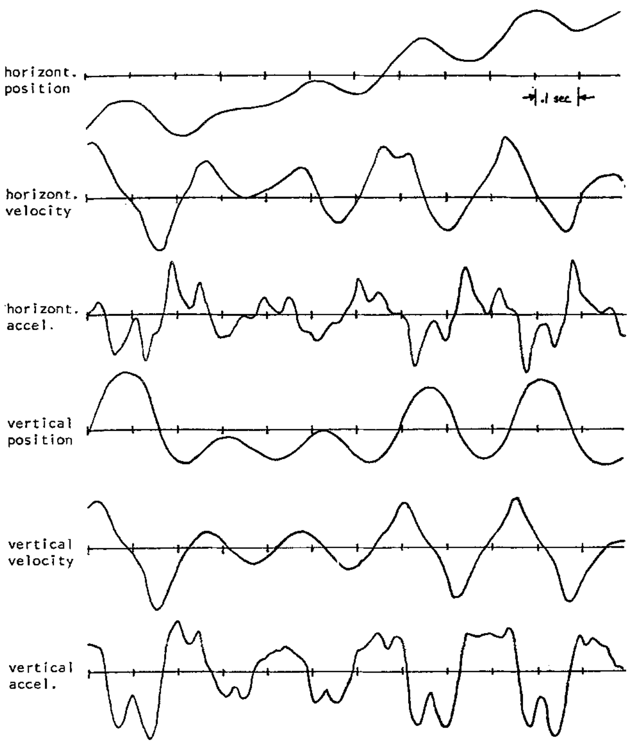


Fig. 20. A complete recording of *hell* by subject KAS

Fig. 21. Acceleration profile for a down-up finger movement on plexiglass by subject HOL

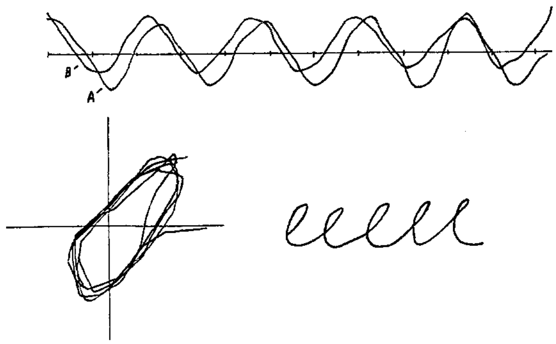


Fig. 22. Recording of a string of *e*'s by subject STE. *Top diagram:* vertical (*A*) and horizontal (*B*) velocity traces. *Bottom left:* velocity space diagram

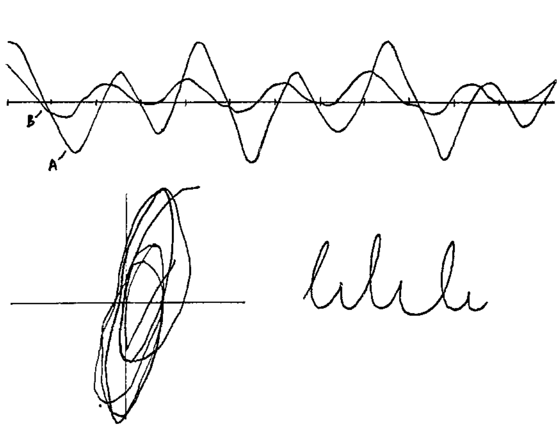


Fig. 23. A recording of *lelele* by subject HOL and the associated velocity space diagram shows constancy of slant, shape, and horizontal sweep velocity. *Top diagram:* vertical (*A*) and horizontal (*B*) velocity traces. *Bottom left:* velocity space diagram

Fig. 24. **A** Signature by subject LAR; **B** signature by subject STE

4.3. *Measurements of Handwriting*

4.3.1. An Underlying Oscillation. The oscillatory nature of human handwriting is illustrated clearly by simple patterns such as chains of *e*'s and *u*'s. A typical example of such writing is presented in Fig. 22. From the velocity traces it is evident that a constant frequency of oscillation is maintained through the extent of this writing. Whether the oscillation is sinusoidal, trapezoidal, or some other pattern is a question examined more closely later.

The frequency of oscillation is a strong function of letter height. Generally speaking, smaller frequencies are associated with greater letter heights. For an alternating pattern of *e*'s and *l*'s, such as the example of Fig. 23, the frequency difference between the *e* and *l* is typically around $\sqrt{2}$. A fair degree of rhythmic constancy is maintained among letters of the same height, i.e., the *e*'s are all made at the same frequency and the *l*'s are all made at the same frequency. The issue of height control is dealt with more thoroughly later.

4.3.2. Horizontal Modulation of Shape. Within a string of letters of the same height, the frequency of oscillation is observed to vary somewhat, although the variation is within bounds of frequencies of letters greater or smaller in height. The oscillation model would lead one to expect that for letters of a given height a constant vertical oscillation frequency is maintained and a modulation of the horizontal oscillation acts independently to shape a letter. A reason for maintaining a constant vertical frequency, it was suggested, was ease of maintaining a straight base line.

To illustrate frequency variation among different letters, two examples of signatures are presented in Fig. 24. In Table 9 the frequencies and letter heights for individual down-up strokes is presented for each signature. Only down-up strokes that return to the same height are considered. In Fig. 24A the strokes from the *a* to the *s* and the stroke *o* are roughly equal in height, and the frequencies range from 4.1 to 3.5. Particularly

Table 9A

Stroke	Frequency (Hz)	Height (mm)
Top of *a*	6.3	0.4
a	4.1	2.7
a to *r*	4.1	2.6
r to *s*	3.8	3.1
o	3.5	2.6
n	4.9	1.2
Bottom of *s*	6.7	0.4

Table 9 B

Stroke	Frequency (Hz)	Height (mm)
e to *v*	5.9	2.4
v	6.3	2.1
v to *e*	5.9	1.6
e to *n*	6.3	1.6
n	5.9	2.0
n to *s*	6.3	1.8

Table 10

Subject	c	$c(\sigma)$	β	$\beta(\sigma)$	zero	zero(σ)
BRU	9.5	3.4	78.2	4.6	−10.8	6.5
	13.6	2.2	70.2	7.9	−17.8	10.3
	12.9	2.1	76.0	7.0	−19.6	4.2
GRI	6.0	1.1	63.6	9.7	−10.8	4.7
	7.5	3.8	65.0	10.2	−12.5	3.6
	9.4	2.1	63.8	6.0	−18.0	4.6
HOL	11.5	1.5	74.5	0.5	− 5.5	2.5
	12.0	3.4	76.3	5.8	−12.7	6.6
	13.3	2.4	79.3	2.3	− 8.7	3.7
LAR	12.4	3.6	62.6	6.8	−12.0	1.7
	12.7	2.5	67.1	2.4	− 8.9	4.3
	16.0	4.5	58.8	2.3	− 9.2	2.6
MAS	12.3	2.3	64.0	8.4	−11.2	4.0
	12.0	1.0	66.0	1.0	− 7.0	0.0
	17.0	1.4	68.0	4.5	− 5.3	4.8
MAT	28.0	6.4	63.7	2.9	−16.3	4.5
	26.3	3.6	64.0	1.6	− 8.5	1.8
	30.2	4.9	63.8	1.5	−14.8	3.8
MCD	19.0	6.3	59.3	8.7	−16.5	9.2
	16.2	3.9	57.2	9.5	−29.0	9.3
	20.5	5.3	54.2	8.2	−27.3	9.8
SJO	10.5	1.3	77.8	3.8	− 8.2	1.6
	16.6	1.6	78.6	3.9	−15.8	5.3
	16.8	2.5	76.4	4.8	−11.6	2.2
STA	11.3	3.0	55.5	6.2	−16.7	7.0
	11.3	1.3	54.5	4.2	−20.0	4.8
	10.8	2.9	56.0	3.3	−16.2	4.4
STE	12.5	2.6	52.3	6.0	−17.8	5.8
	6.5	4.8	69.0	4.6	−13.2	2.3
	11.8	3.3	66.0	3.6	−16.8	5.0
WOO	11.0	3.7	67.5	3.2	− 5.7	2.2
	11.4	1.5	65.8	5.2	− 6.8	6.4
	13.8	6.2	66.5	5.2	− 6.2	3.5

for the *o* there has been some modulation of vertical frequency, which shows that a vertical modulation participates with the horizontal modulation for letter shape. The down-up stroke for the *n* is half the height of the previous strokes and has a correspondingly higher frequency 4.9. Similarly the clockwise down-up stroke at the top of the *a* and the up-down stroke at the bottom of the *s*, both of very small height, are executed at the high frequencies 6.3 and 6.7, respectively. The second signature in Fig. 24B shows a greater degree of rhythmic regularity than the previous example. The range of frequencies is tight, between 5.9 and 6.3. Curiously the height variation is greater than the frequency variation.

4.3.3. Constant Horizontal Velocity Sweep. In the earlier mathematical treatment it was assumed that the writing movement could be factored into an oscillatory movement superimposed on a constant horizontal velocity sweep. This assumption was substantially validated by measurements on human writing. Subjects were asked to produce strings of letters *lelele.* The linear sweep for a given letter in the string was estimated by fitting a least squares ellipse to the velocity space diagram. The mean value for the constant velocity sweep c and the standard deviation $c(\sigma)$ for the letters within one of these strings was tabulated in Table 10 for a number of subjects and for 3 trials for each subject; the units are cm/s.

The standard deviations show a fair degree of constancy of c from letter to letter within a given string. Some subjects showed a greater degree of constancy than others. The same observations apply between writing samples for a given subject. The standard deviations are actually less significant than might appear because the horizontal velocity amplitudes, not indicated in Table 19, are typically around 3 times greater than the constant velocity sweep.

4.3.4. Height Control. Earlier two possible strategies for height control were examined: modulation of amplitude and modulation of frequency. Actual experimental measurements indicate that both modulations take place. Subjects were asked to produce a 6 letter sequence of alternating *e*'s and *l*'s. Measuring each letter from bottom corner to bottom corner, the ratios of *l* to *e* letter duration, vertical velocity amplitude, and height were calculated and averaged over 3 trials (Table 11). The average over all subjects is also given. The results show that vertical velocity amplitude is approximately double for the *l* as compared to the *e*. This doubled amplitude is obtained in roughly equal measure from a frequency modulation roughly proportional to $\sqrt{2}$ and an acceleration amplitude modulation roughly proportional to $\sqrt{2}$. Because of the double integration of acceleration the proportional contribution to height for frequency vs. amplitude modulation is $2 : \sqrt{2}$.

The frequency modulation is not a result of limitations in the power plant. Muscles involved in handwriting seem grossly overpowered for the task. There are two pieces of evidence for this assertion. 1) As mentioned earlier handwriting frequency was not affected when weights were attached to the accelerometer apparatus. 2) Handwriting frequency is independent of writing size. This independence is an accepted observation in the handwriting literature (Denier van der Gon and Thuring, 1965; Yasuhara, 1975) and is an observation substantiated by my own measurements. Subjects were asked to produce the alternating *el* patterns in different sizes. The *l* height and the time required to produce one *el* pair were calculated and averaged over 3 trials (Table 12). Table 12 indicates that the amount of time required to write an *el* pair is independent of size over a factor of 2 across a sampling of subjects.

In Fig. 25 the vertical acceleration profile for two different sizes of the same word, one twice the size of the other, nearly overlap. Thus the handwriting muscles are capable of writing tall letters in the same time as short letters. Yet an *e* in large writing similar in shape but taller than an *l* in small writing is written in less time, in fact in precisely the same time as an *e* in the small writing.

The frequency modulation by a factor of $\sqrt{2}$ fits particularly well the alternate activation mode of the spring model. Presumably an extra amplitude modulation would be required because of stylistic constraints to produce a sufficiently tall *l*. This magnitude of frequency modulation is also well within the range of a variable spring constant model.

4.3.5. Slant Constancy. The extent to which subjects maintained constant slant during writing was determined by computing the slant of individual letters in the sequence *lelele*. The slant of a letter was computed by fitting a least squares ellipse to the velocity space diagram corresponding to that letter and by determining the slant from the coefficients of this ellipse. These measurements are model based in the sense that a sinusoidal oscillation is presumed to underly the experimental velocity space diagram. The results in Table 10 under the column β and $\beta(\sigma)$ are expressed as a mean value of slant and standard deviation for the letters in one sequence *lelele*.

Table 10 shows that there are mixed results with regard to consistency of writing slant. For consistency of slant within a single word or letter sequence, some subjects during certain trials had highly consistent writing slants with a standard deviation below 4°. At the other extreme some subjects exhibited poor slant control with standard deviations above 8°. Most subjects fell more or less between these extremes. For consistency of slant between trials, the results are also mixed. Generally the range of slant means between trials did not exceed 10°, with some subjects having a range of 5° or less. Finally, although this breakdown is not given in Table 10, the slants of *e*'s were more or less the same as the slants of neighboring *l*'s, with a tendency in some subjects to make the *e*'s more slanted.

It can be concluded that generally speaking people are moderately successful at maintaining a consistent writing slant.

Table 11

Subject	Duration ratio	Velocity ratio	Height ratio
BRU	1.39	1.75	2.25
GRI	1.25	1.97	2.46
HOL	1.39	2.11	2.92
LAR	1.39	1.99	2.59
MAS	1.52	2.05	2.71
MAT	1.40	1.81	2.36
SJO	1.34	2.06	2.52
STE	1.44	2.34	2.98
ULL	1.26	1.85	2.12
GROUP	1.38	1.99	2.55

Table 12

Subject	*l* height (in)	*el* time (ms)
BRU	0.38 in	445
	0.86	435
MAT	0.34	547
	0.46	558
SJO	0.20	403
	0.31	423
STE	0.25	522
	0.37	503
	0.49	498

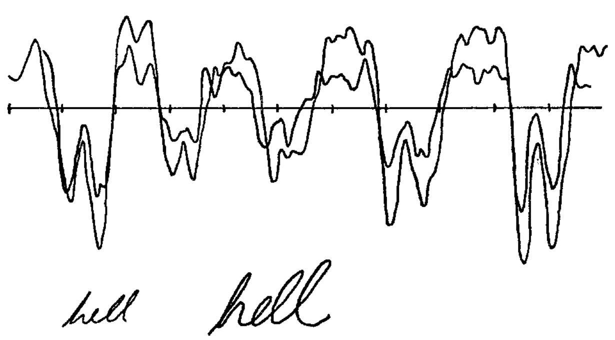


Fig. 25. Vertical acceleration profiles from two different writing sizes of *hell*, one double the other, by subject STE show a high degree of burst duration agreement

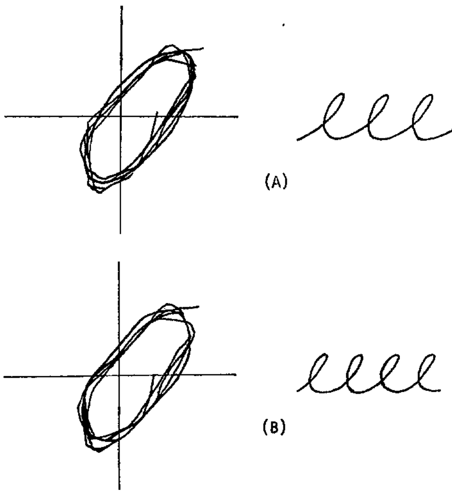


Fig. 26. **A** A best fit ellipse and the resultant writing. **B** A best fit rounded parallelogram and the resultant writing

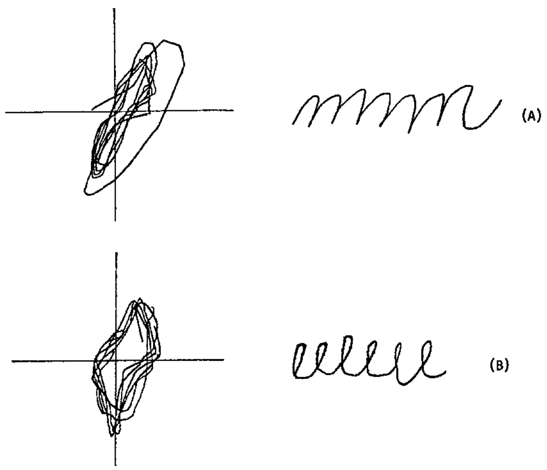


Fig. 27A and B. By phase shifting the horizontal velocity pattern in **A**, the parallelogram nature of the velocity space diagram **B** emerges

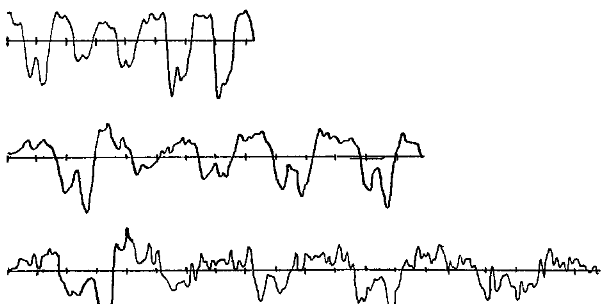


Fig. 28. Vertical acceleration profiles from three samples of *hell* written at different speeds show trapezoidal bursts

4.3.6. Shape Constancy. It was suggested earlier that shape control at top corners is related to the horizontal velocity at the top vertical velocity zero crossing. To what extent is shape constancy maintained in going between *e*'s and *l*'s? In Table 10 the mean horizontal velocity at the vertical velocity zero crossing, labeled "zero" in Table 10, and the associated standard deviation are presented for the sequence *lelele*. As opposed to the previous columns in Table 10, the zero values are direct measurements rather than extrapolated values from a least squares ellipse; the units are cm/s.

A rough shape constancy is demonstrated by Table 10. Thus subjects seem to attempt to keep the same top shape when going between *e*'s and *l*'s.

4.3.7. The Forcing Function. A major advantage of the spring muscle model is that a sinusoidal oscillation can be obtained with minimal effort. Starting from an appropriate set of initial conditions, a sinusoidal oscillation propagates indefinitely with quite passive control; i.e., the zero settings are not changed. The main conclusions however are not dependent on the validity of the spring muscle model. A sinusoidal forcing function arising from active central programming would also be subject to the same slant, shape, and height constraints as a sinusoidal function arising from the spring model. Only the details of the modulations would differ.

In terms of the position or velocity recordings for writing of a given height, the fit of the spring model and of the various flavors of step patterns is about equally good. Actual velocity space diagrams are ambiguous with regard to being ellipsoidal or rounded parallelogram. Figure 26 shows best fit ellipse and a best fit rounded parallelogram to Fig. 22 and the associated synthetic writing. Both agree fairly well with the data. Less ambiguous is the diagram in Fig. 27A, harboring a parallelogram hidden by a nearly collapsed diagram. Shifting the horizontal velocity, which does not change the nature of the velocity space diagram, brings out the underlying parallelogram (Fig. 27B).

With regard to recorded accelerations, trapezoidal and sinusoidal patterns seem to fit the data for fast writing about equally well. With progressively slower writing, however, large acceleration plateaus emerge (Fig. 28). Assuming the underlying control mechanism does not change with writing speed, an assumption which seems born out by the similarity of the acceleration profiles, the fast writing bursts would seem to have a trapezoidal basis.

The sharp negative acceleration peaks, present in Fig. 28 and in most other writing data, are foreign to both patterns, however. As indicated in Sect. 4.1, these peaks are not just friction artifacts. They may indicate a segmentation of the acceleration profiles at the top corners; although the underlying pattern is continuous, it might be thought of as a chain of down-up strokes. Some writing specimens may show more than

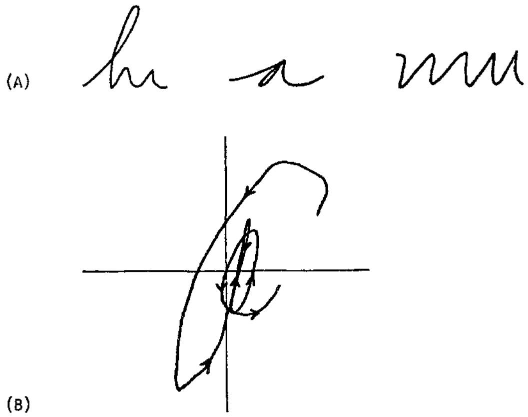


Fig. 29. A The letters *h* in *hi*, *a*, and *m* in *mu* are examples of modified letter forms by three different fast writers. **B** The velocity space diagram for the *hi* shows only counterclockwise movement

two peaks per burst, however, and may show peaks in the positive bursts as well.

4.3.8. Clockwise Elimination. There are a number of modifications and shortcuts taken by fast writers to standard cursive script shapes in order to adapt the shapes to requirements of speed and rhythmicity. In Fig. 29 the *h* in *hi*, the letter *a*, and the *m* in *mu* are examples of modified shapes produced by three different fast writing subjects. The common feature of these letters in the Palmer script is a clockwise movement: to round the top of the *a*, and to produce the bottom cusp corners of *h* and *m*. In examining the velocity space diagrams for these letters by fast writing subjects, for example that of *hi* in Fig. 29B, what stands out is that clockwise movement has been completely eliminated in favor of a uniform counterclockwise movement. At most a straight line velocity space diagram is obtained, corresponding to zero phase difference between horizontal and vertical joints (Fig. 2D); that is to say, clockwise shapes are approximated with an essentially rounded sawtooth pattern.

5. Conclusions

The oscillation theory of handwriting suggests that letter shapes emerge as individuations of an underlying oscillation. The alternative view challenged by this theory is that each letter has a separate motor program which can be invoked to produce that letter in isolation, and that the word formation process is one of linking together the motor programs for the desired letters. In the oscillation theory there is a preexisting and underlying repeated pattern of letter shapes, for example a cycloidal chain of *e*'s for a sinusoidal based oscillation, and that this pattern propagates indefinitely unless it is modulated. Rather than an active process of forming letter shapes, there already exist letter shapes typical of the oscillation pattern and the modulations serve to remold the preexisting letter shapes into the desired letters. A modulation will change the underlying oscillation pattern to a new one, which like the old will propagate indefinitely unless it too is modulated. In a sinusoidal based oscillation, for example, an original *e* cycloid can be modulated to an *l* cycloid, and after this modulation the new underlying pattern is the *l* cycloid.

In motor control work one is used to thinking in terms of motor programs, and here the motor programs are best considered the sequence of modulations. The underlying oscillatory process acts as an interpretive program that "interprets" the motor programs, which are the sequence of modulations, in the context of the current oscillation.

In creating a word a temporal sequence of modulations to some oscillation pattern must be set up. Even for writing single isolating letters such as an *a* an oscillation must be created and one or two modulations applied to it. There is a question as to the size and the nature of the conceptual unit in handwriting. Is the conceptual unit linguistically based, such as a syllable or a word, or is it based on some oscillation feature, such as whether the movement is clockwise or counterclockwise or whether adjacent letters have the same height? What the conceptual unit in handwriting is and whether it too should be considered a single motor program remains an open question.

The human motor system has configured itself to make the handwriting act a relatively simple task. It has factored the large numbers of degrees of freedom of hand and arm into just a few degrees of freedom that facilitate the control of handwriting. Coupled oscillations in x and y directions produce diverse corner shapes; a potential third degree of freedom for a horizontal constant velocity movement separates the corners to form letters and words. The process of letter shaping reduces largely to controlling the vertical velocity zero crossing in the velocity space diagram: the intercept controls corner shape while the slope at the zero crossing controls writing slant. Under the constraints of the zero crossings letter height is modulated by both frequency and amplitude modulation of the acceleration.

The oscillation-modulation scheme reduces the information processing requirements for handwriting at the expense perhaps of letter shape diversity. It might be speculated that a reduction of the information processing complexity for handwriting is necessary for thinking and writing at the same time.

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References

Abelson, H., di Sessa, A., Rudolph, L.: Velocity space and the geometry of planetary orbits. Am. J. Phys. **43**, 579–589 (1975)

Bizzi, E., Polit, A., Morasso, P.: Mechanisms underlying achievement of final head position. J. Neurophys. **39**, 435–444 (1976)

Crane, H.D., Savoie, R.E.: An on-line data entry system for hand-printed characters. Computer, March 1977, 43–50 (1977)

Denier van der Gon, J.J., Thuring, J.Ph.: The guiding of human writing movements. Kybernetik **2**, 145–148 (1965)

Eden, M.: Handwriting generation and recognition. In: Recognizing patterns, Kolers, P.A., Eden, M. (eds.), pp. 138–154. Cambridge, MA: MIT Press 1968

Feldman, A.G.: Change of muscle length due to shift of the equilibrium point of the muscle-load system. Biofizika **19**, 534–538 (1974a)

Feldman, A.G.: Control of muscle length. Biofizika **19**, 749–753 (1974b)

Grillner, S.: Locomotion in vertebrates: central mechanisms and reflex interaction. Physiol. Rev. **55**, 247–304 (1975)

Hagan, P.: Johnny not only can't write: he can't handwrite, either. N.Y. Times Sunday Magazine, May 23, 64 (1976)

Herbst, N.M., Liu, C.N.: Automatic signature verification based on accelerometry. IBM J. Res. Develop., May 1977, 245–253 (1977)

Hertzberg, O.E.: A comparative study of different methods used in teaching beginners to write, No. 214. Teachers College, Columbia University 1926

Koster, W.G., Vredenbregt, J.: Analysis and synthesis of handwriting. In: Biomechanics. II, p. 77. (Medicine and Sport, Vol. 6.) Bern, Stuttgart, Wien: Hans Huber 1977

Lawerence, J.D.: A catalog of special plane curves. Dover: Dover Publ. 1972

McDonald, J.S.: Experimental studies of handwriting signals. RLE Technical Report 443. Cambridge, MA: MIT Press 1966

Mermelstein, P.: Computer recognition of connected handwritten words. PhD Thesis, Massachusetts Institute of Technology 1964

Mermelstein, P., Eden, M.: Experiments on computer recognition of connected handwritten words. Inf. Control **7**, 255–270 (1964)

Paul, R.: Modelling trajectory calculation and servoing of a computer controlled arm. Stanford Artificial Intelligence Memo 72, 1972

Pellionisz, A., Llinas, R.: Brain modeling by tensor network theory and computer simulation. The cerebellum: distributed processor for predictive coordination. Neuroscience **4**, 324–348 (1979)

Rack, P.M.H., Westbury, D.R.: The effects of length and stimulus rate on tension in the isometric cat soleus muscle. J. Physiol. (London) **204**, 443–460 (1969)

Raibert, M.H.: Motor control and learning by the state space model. AI-TR-439, MIT Artificial Intelligence Laboratory 1977

Yasuhara, M.: Experimental studies of handwriting process. Rep. Univ. Electro-Comm. 25-2 (Sci. and Tech. Sect.), 233–254 (1975)

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