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The generation of handwriting with delta-lognormal synergies

Réjean Plamondon, Wacef Guerfali

École Polytechnique de Montréal, Laboratoire Scribens, Département de génie électrique et de génie informatique, C.P. 6079, Succursale Centre-Ville, Montréal QC H3C 3A7, Canada

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Abstract. This paper presents a handwriting generation model that takes advantage of the asymptotic impulse response of neuromuscular networks to produce and control complex two-dimensional synergistic movements. A parametric definition of a ballistic stroke in the context of the kinematic theory of rapid human movements is given. Two types of parameters are used: command and system parameters. The first group provides a representation of the action plan while the second takes into account the temporal properties of the neuromuscular systems executing that plan. Handwriting is described as the time superimposition of basic discontinuous strokes that results in a continuous summation of delta-lognormal velocity vectors. The model leads to trajectory reconstruction, both in the spatial and in the kinematic domain. According to this new paradigm, the angular velocity does not have to be controlled independently and continuously; it naturally emerges from the vectorial summation process. Several psychophysical phenomena related to two-dimensional movements are explained and analyzed in the context of the model: the speed/accuracy tradeoffs, spatial scaling, the isochrony principle, the two-thirds power law, effector independence, etc. The overall approach also shows how basic handwriting characteristics (dimension, slant, baseline, shape, etc.) are affected and controlled using an action plan made up of virtual targets fed into a neuromuscular synergy that is governed by a delta-lognormal law.

1 Introduction

Handwriting stands among the most complex tasks performed by literate human beings. It requires sensory-motor control mechanisms involving a large spectrum of cerebral activities, dealing with the emotions, rational thought and communication. As such, the study of handwriting constitutes a very broad field that allows researchers with various backgrounds and interests to collaborate and interact at multiple levels with different but complementary objectives (see, for example, van Galen et al. 1991; Wann et al. 1991; van Galen and Stelmach 1993; Plamondon 1993a, 1994, 1995a; Faure et al. 1994; Simner et al. 1996). In this paper, we are interested in understanding handwriting generation at the global neuromuscular level, focusing mainly on the development of a stroke generation model that is general enough to explain the origin of some basic psychophysical laws of simple human movements and to show how human subjects can take advantage of the knowledge that emerges from the properties of this representation to control the sensory-motor interactions involved in the generation of more complex trajectories. The overall approach is based upon the hypothesis that complex human movements are made up of, and can be segmented into, basic and simple units. In other words, due to the intrinsic properties of the neuromuscular system involved in a rapid writing task, there is a class of simple movements, hereafter called strokes, that are preferentially produced by such a system, once it is well-trained. More complex movements are thus generated by the vectorial addition of the various strokes belonging to such a class.

The concept of a ballistic or a fundamental unit of human movement is not new. It originated in the Lashley experiments (Lashley 1917) and was adapted to the analysis of handwriting in the 1960s (Eden 1962; Denier van der Gon and Thuring 1965, etc.). Similarly, the idea of overlapping instead of concatenating discontinuous strokes to study handwriting was put forward by Morasso and Mussa Ivaldi (1982) in the early 1980s. These concepts have been among the cornerstones upon which numerous handwriting generation models have been built (see Plamondon and Maarse 1989, for a review of the models published prior to 1989; Schomaker et al. 1989; Bullock et al. 1993; Morasso and Sanguinetti 1993; Morasso et al. 1994; Stettiner and Chazan 1994; Singer and Tishby 1994).

Although these models have been developed for different purposes with different practical or theoretical goals in mind, the majority of their authors were aiming at generating human-like handwriting. When a systematic comparison of these models is performed using real handwritten data, it is clear that some models perform better than others (Plamondon et al. 1993; Alimi and Plamondon 1993, 1994). This does not mean, however, that a model cannot be useful for a specific application, if its output is not as realistic as that of another model. The decision to use a model always depends on the goal of the research project. For example, the design requirements are very different if one is developing an interactive system to help children learn handwriting or if one is interested in segmenting letters for pattern recognition purposes.

In this paper, our objective is to use the kinematic theory of rapid human movements (Plamondon 1993c,d, 1995b,c, 1996, 1997a) to analyze and understand handwriting generation and control. The kinematic theory describes the global properties of the neuromuscular networks involved in a synergistic action. In a single framework, it explains the origin of the basic kinematic relationships and psychophysical laws that have been consistently reported in studies dealing with rapid human movements (Plamondon and Alimi 1996; Plamondon 1997b). This paper shows how the same approach can be used and generalized to study and to understand the basic properties of pentip trajectories as produced by human subjects performing a writing task (Plamondon 1995d; Plamondon and Guerfali 1996b).

This article deals with three major themes. First, in Sect. 2, we summarize the neuromuscular model that emerges from the kinematic theory and use it to describe the basic properties of a single stroke. Then, in Sect. 3, the generation of handwriting is studied in the context of a vectorial delta-lognormal model (Plamondon 1995d). Multi-stroke movements like letter and word production are investigated (Guerfali and Plamondon 1995a, b; Guerfali 1996). Theoretical and practical considerations concerning the quantitative use of this new model to analyze and synthesize handwriting are also examined. Finally, in Sects. 4 and 5, some basic psychophysical properties often reported in handwriting studies are predicted using computer simulations. In addition, the global perspective provided by this new approach is compared with those of other models previously published.

2 Stroke generation model

Figure 1a schematizes the global features of the stroke generation model that results from the kinematic theory (Plamondon 1995d). According to this scheme, a stroke is produced by the synergistic control of the pentip velocity $\vec{v}(t)$. This is achieved by synchronously activating two global neuromuscular systems, one agonist and the other antagonist, that control the end-effector movement around a generalized joint. Each system has an impulse response, asymptotically described by a lognormal function $\Lambda(t; t_0, \mu, \sigma^2)$ (Plamondon 1991, 1993b, 1995b,c). These two systems constitute a neuromuscular synergy that produces, from a given initial posture P_0, a stroke along a circular path of curvature C_0 around the generalized joint, in a given initial direction θ_0. To execute this basic movement, two input commands $\vec{D}_{1(P_0,\theta_0,C_0)}U_0(t-t_0)$ and $\vec{D}_{2(P_0,\theta_0,C_0)}U_0(t-t_0)$ are fed simultaneously into the agonist and antagonist systems, at a given time t_0. Each of these two systems reacts to its specific input with a lognormal impulse response in a logtime delay μ_1 or μ_2, and with a logresponse time of σ_1 or σ_2 respectively (Plamondon 1993c, 1995d).[1]

In this context, a stroke executed from an arbitrary starting position P_0 is characterized by nine parameters. At the action plan level, C_0 and θ_0 reflect the global geometric properties of the set of muscles and joints recruited to execute the movement, whereas the parameters D_1, D_2 and t_0 provide a synthetic description of the input commands. The parameters μ_1, μ_2, σ_1 and σ_2 describe the global timing properties of the neuromuscular networks involved in generating the movement. In other words, the different parameters take a single specific value for a given stroke, but, in practice, these parameters can be considered as random variables with specific distributions. Analyzing a set of similar strokes should highlight the statistical properties of these distributions.

The key feature of a single stroke is that the magnitude of its velocity $|\vec{v}(t)|$, often referred to as curvilinear velocity, along the circular path is described by (Plamondon 1993c, 1995b):

$$|\vec{v}(t)| = \Big|\vec{D}_{1(P_0,\theta_0,C_0)}\Lambda(t; t_0, \mu_1, \sigma_1^2) - \vec{D}_{2(P_0,\theta_0,C_0)}\Lambda(t; t_0, \mu_2, \sigma_2^2)\Big| \tag{1}$$

where

$$\Lambda(t; t_0, \mu_j \sigma_j^2) = \frac{1}{\sigma_j\sqrt{2\pi}(t-t_0)} \exp\left(\frac{-[\ln(t-t_0)-\mu_j]^2}{2\sigma_j^2}\right) \tag{2}$$

with $t > t_0$.

This latter expression has been shown to be the best descriptor of individual stroke velocity profiles out of 26 different models tested over the same database (Plamondon et al. 1993; Alimi and Plamondon 1993, 1994). It also incorporates in its formulation all the global characteristics of rapid movements, from basic kinematic and kinetic relationships to speed/accuracy trade-offs (Plamondon 1993c,d, 1995b,c, 1996, 1997a; Plamondon and Alimi 1997; Plamondon 1997b).

The direction of the stroke velocity ($\angle(\vec{v}(t))$ with respect to any arbitrary reference can be described as a function of time by:

$$\angle\vec{v}(t) = \theta(t) = \theta_0 + C_0 \int_{t_0}^{t} |\vec{v}(\tau)| d\tau \tag{3}$$

and its first derivative as a function of time (often referred to as the angular velocity (Plamondon 1987, 1989a, 1992) gives

$$\nu_\theta(t) = \frac{d\theta(t)}{dt} = C_0|\vec{v}(t)| \tag{4}$$

This latter expression (4) shows that, for a single circular stroke, the amplitude of the angular velocity also has an 'asymmetric bell-shaped velocity profile' described by a delta-lognormal law. This is consistent with the data reported

[1] The parameters μ and σ are temporal parameters that represent the global time delay and the response time of a neuromuscular network on a logarithmic time scale. We use the terms logtime delay and logresponse time to point out this scaling effect (Plamondon 1993c, 1995b, 1997b).

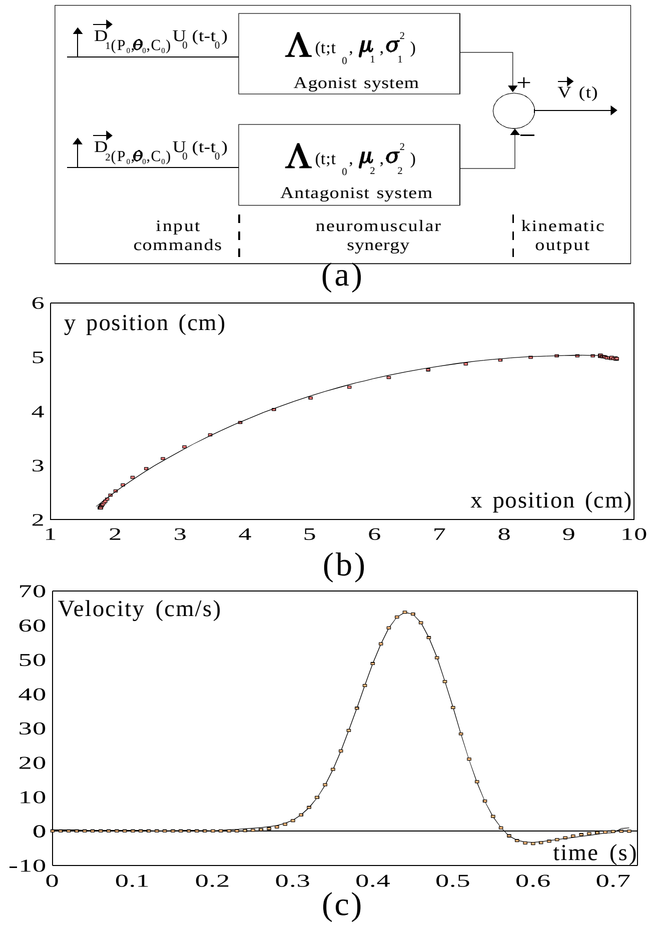

Fig. 1. a Kinematic description of a synergy. **b** Single stroke: *continuous line*, original data; *squares*, model predictions. **c** Signal velocity profile (cm/s): *continuous line*, original data; *squares*, model predictions

in various studies dealing with specific rotation movement along a single joint (see, for example, Morasso 1981; Abend et al. 1982).

Figure 1b shows a typical large stroke and Fig. 1c the magnitude of its velocity vector. The velocity signal presented in Fig. 1c was plotted using a bipolar or signed representation to highlight the change of direction associated with the small oscillations occurring at the end of the movement. On each graph, the predictions of the kinematic theory are reported using squares superimposed on the continuous lines that extrapolate the original digitizer data, sampled at 100 Hz. As can be seen, the model in Fig. 1a is able to reconstruct a stroke, both in the image and in the kinematic domain. Similar results have been reported over different databases for more than 3000 strokes in all (Alimi and Plamondon 1993; Alimi 1995).

Another important feature of the kinematic theory is the analytical equation predicting the duration of a single stroke with respect to its relative spatial accuracy (Plamondon 1993d, 1995c). In short, the theory demonstrates that, for a specific group of neuromuscular networks recruited for executing a specific stroke, it is possible for a human subject to anticipate the duration of that stroke with respect to the relative spatial precision of the target that he or she is aiming at. In its simplest form,[2] the kinematic theory predicts that

$$MT = K \left(\frac{D}{\Delta D} \right)^{\alpha} \tag{5}$$

where MT is stroke duration, D is stroke amplitude, with $D = |\vec{D}_{1(P_0,\theta_0,C_0)} - \vec{D}_{2(P_0,\theta_0,C_0)}|$, ΔD is absolute error of the stroke amplitude, and K and α are constants depending on μ_j, σ_j.

As will be seen below, this prediction is of major importance when it comes to understanding the generation of multiple-stroke movements. This speed/accuracy trade-off shown in (5) provides a framework that can be used as a possible explanation for the capacity of human subjects to anticipate and superimpose strokes to generate fluent and more complex trajectories. Indeed, the kinematic theory leads to the assumption that once a stroke has been initiated, that is, once $\vec{D}_{1(P_0,\theta_0,C_0)}$ $(U_0(t-t_0)$ and $\vec{D}_{2(P_0,\theta_0,C_0)}$ $U_0(t-t_0)$ start simultaneously activating the proper neuromuscular systems, a well-trained subject can know in advance that the target, a stroke of amplitude D, will be executed with an absolute error $\pm\Delta D$, within a movement time MT. The next stroke can thus be initiated before the completion of the current one, as though this latter stroke had been completed and its target had been reached (Plamondon 1995d).

3 Handwriting generation: The vectorial delta-lognormal model

From this perspective, the production of fluent handwriting can be seen as the vectorial superimposition in time of different strokes. The image of a trajectory simply results from this vectorial summation process. In terms of differential geometry, a continuous handwritten trace is considered as a plane curve, described by its curvature along the curvilinear axis ξ:

$$C(\xi) = \frac{d\theta(\xi)}{d\xi} \tag{6}$$

where $\theta(\xi)$ represents the angular direction along the arc length.

According to the vectorial delta-lognormal model, a trace is memorized as a symbolic action plan representing a sequence of virtual targets to be linked with strokes in a two-dimensional (2D) space to produce a letter or word. Starting from that action plan, a movement-sequencing mechanism generates the proper series of input commands to the system depicted in Fig. 1a (Plamondon and Privitera 1995; Privitera and Plamondon 1995). The resulting velocity $\vec{\nu}(t)$ of the pentip, for a sequence of strokes linking n virtual targets, is thus obtained by summing the vectors representing each individual stroke velocity:

$$\vec{\nu}(t) = \sum_{i=1}^{n} \vec{\nu}_i(t - t_{0i}) \tag{7}$$

where each $\vec{\nu}_i(t-t_{0i})$ is described by (1) and (3), for $t > t_{0i}$.

[2] A more complex formulation, in terms of a quadratic law, describes the general case (Plamondon 1993d, 1995c).

The trajectory is planned from a given position P_0 without any reference to a specific axis system and the action plan is invariant to rotation. The global orientation of the handwritten trace is embedded in the sequence of θ_{0i} values. Depending on these geometric constraints, the neuromuscular systems recruited for each stroke will respond more or less rapidly to the input commands, as reflected by the corresponding values of the temporal parameters (μ_j, σ_j) of the systems.

If the vectorial summation of n strokes is analyzed and studied in a specific cartesian reference system (as defined, for example, by a digitizer), the instantaneous magnitude and direction of the velocity $\vec{\nu}(t)$ can be described by:

$$|\vec{\nu}(t)| = \left\{ \left[\sum_{i=1}^{n} \nu_{xi}(t - t_{0i}) \right]^2 + \left[\sum_{i=1}^{n} \nu_{yi}(t - t_{0i}) \right]^2 \right\}^{1/2} \tag{8}$$

and

$$\theta(t) = \text{arctg} \left\{ \frac{\sum_{i=1}^{n} \nu_{yi}(t - t_{0i})}{\sum_{i=1}^{n} \nu_{xi}(t - t_{0i})} \right\} \tag{9}$$

where

$$\nu_{xi}(t - t_{0i}) = \nu_i(t - t_{0i}) \cos(\theta_i(t - t_{0i})) \tag{10}$$

$$\nu_{yi}(t - t_{0i}) = \nu_i(t - t_{0i}) \sin(\theta_i(t - t_{0i})) \tag{11}$$

and $\theta_i(t - t_{0i})$ is defined by (3).

For the simple case where only two strokes are activated at times t_{01} and t_{02} respectively, (7) and (8) reduce to

$$|\vec{\nu}(t)| = \sqrt{\nu_1^2(t - t_{01}) - \nu_2^2(t - t_{02}) + 2\nu_1(t - t_{01})\nu_2(t - t_{02}) \cos(\Delta\theta(t))} \tag{12}$$

where

$$\Delta\theta(t) = \theta_1(t - t_{01}) - \theta_2(t - t_{02}) \tag{13}$$

and

$$\angle\vec{\nu}(t) = \theta(t) = \text{arctg} \left\{ \frac{\nu_1(t-t_{01}) \sin(\theta_1(t-t_{01})) + \nu_2(t-t_{02}) \sin(\theta_2(t-t_{02}))}{\nu_1(t-t_{01}) \cos(\theta_1(t-t_{01})) + \nu_2(t-t_{02}) \cos(\theta_2(t-t_{02}))} \right\} \tag{14}$$

$$= \text{arctg} \left(\frac{f(t)}{g(t)} \right) \tag{15}$$

As can be seen, even in this simple case, the angular velocity $\nu_\theta(t)$, as obtained from the time derivative of $\theta(t)$, is a rather complex analytical formula linking the specific cartesian components of each individual stroke velocity:

$$\nu_\theta(t) = \frac{\frac{df(t)}{dt} g(t) - f(t) \frac{dg(t)}{dt}}{f^2(t) + g^2(t)} \tag{16}$$

where

$$\begin{aligned} \frac{df(t)}{dt} = {} & \frac{d\nu_1(t - t_{01})}{dt} \sin(\theta_1(t - t_{01})) \\ & + C_{01}\nu_1^2(t - t_{01}) \cos(\theta_1(t - t_{01})) \\ & + \frac{d\nu_2(t - t_{02})}{dt} \sin(\theta_2(t - t_{02})) \\ & + C_{02}\nu_2^2(t - t_{02}) \cos(\theta_2(t - t_{02})) \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{dg(t)}{dt} = {} & \frac{d\nu_1(t - t_{01})}{dt} \cos(\theta_1(t - t_{01})) \\ & - C_{01}\nu_1^2(t - t_{01}) \sin(\theta_1(t - t_{01})) \\ & + \frac{d\nu_2(t - t_{02})}{dt} \cos(\theta_2(t - t_{02})) \\ & - C_{02}\nu_2^2(t - t_{02}) \sin(\theta_2(t - t_{02})) \end{aligned} \tag{18}$$

Using (12), (16) and (6) for each stroke, the complete kinematics of a 2D trajectory can be recovered, as can the pentip position with respect to any arbitrary postural reference P_0. Figure 2 shows how the French word ‘elle’ (‘she’ in English), as generated by a human subject, can be reconstructed using the vectorial delta-lognormal model. Here again, squares represent model predictions, and the continuous lines the interpolation of real data, sampled at 100 Hz. As can be seen, the vectorial summation of time-superimposed strokes provides a realistic description of both the visual and the kinematic aspects of a word. In this example, the movement starts at rest ($|\vec{\nu}(t)| = 0$) and the complex pattern of the curvilinear velocity can be accounted for by adding delta-lognormal velocity patterns (12). The angular velocity as predicted by (16) shows a similar oscillatory pattern, with a phase shift of about 180° with respect to $|\vec{\nu}(t)|$. This phase shift between angular and curvilinear velocity is implicit in the model, and is depicted in Figs. 3 and 4 as well. It simply emerges from the vectorial superimposition of discontinuous strokes. This phenomenon has been reported for more than a century (Binet and Courtier 1893; Jack 1895) and been studied by numerous groups. Moreover, contrary to what has been assumed in the preliminary version of our model (Plamondon 1987, 1989a, 1991, 1992), it is clear that continuous control of the angular velocity is not necessary to produce fluent handwriting (Plamondon 1995d; Guerfali 1996).

There are several ways to perform the analysis-by-synthesis of a given handwritten trace (Plamondon and Guerfali 1996b; Ménier et al. 1997; Guerfali and Plamondon 1997). In this paper, we have used the following algorithm (Guerfali 1996):

- compute the module and the direction of the curvilinear velocity from the $X(t)$, $Y(t)$ coordinates of a word written on a digitizer,
- using the pen-paper contact information available from the digitizer, perform a first-level segmentation of the word into components, defined as the pentip traces produced during a continuous pen-down signal,
- for each component, extract the partially hidden strokes by optimizing the matching of the curvilinear velocity, with a sum of delta-lognormal equations delayed in time, as well as matching the angular velocity curve that results from this vectorial addition of strokes,
- save the nine parameters that best represent each stroke.

Each handwritten component is thus made up of a set of *strokes* that are superimposed in time to produce a fluent trace. According to the model, these strokes constitute the basic units of human writing movements and serve as the coding elements in the motor planning of complex trajectories. Each stroke is characterized by a velocity vector $\vec{\nu}(t)$ whose magnitude obeys the delta-lognormal law (1). A stroke is thus an arc of a circle characterized by nine param-

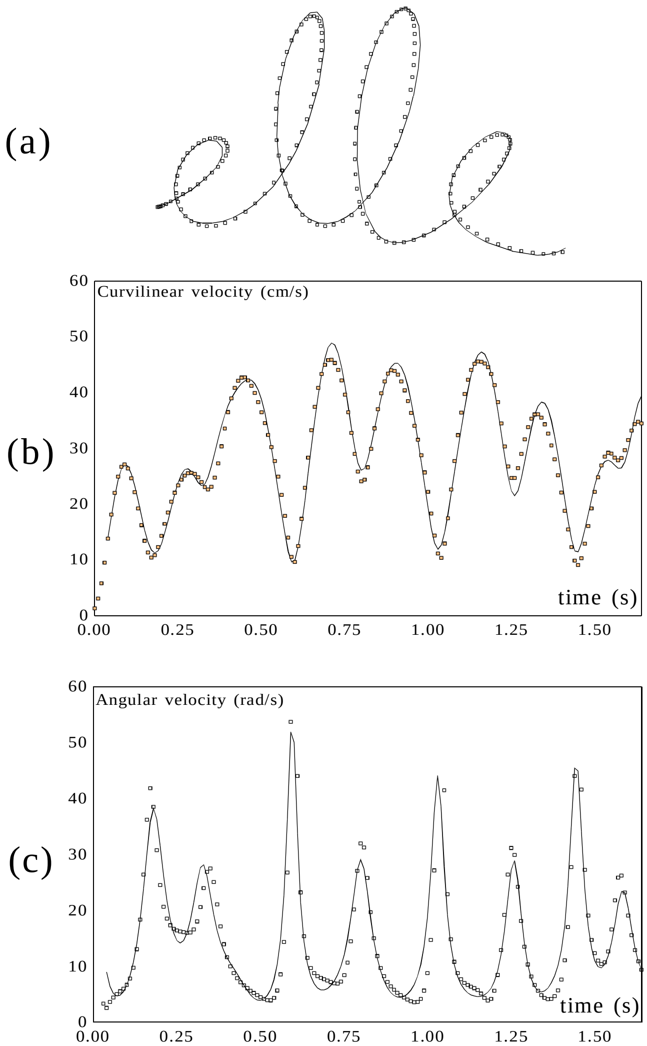

Fig. 2. a Handwriting specimen (French word 'elle'): *continuous line*, original data; *squares*, model predictions. **b** Curvilinear velocity (cm/s): *continuous line*, original data; *squares*, model predictions. **c** Angular velocity (rad/s): *continuous line*, original data; *squares*, model predictions

eters: C_0, θ_0, t_0, D_1, D_2, μ_1, μ_2, σ_1, σ_2. These strokes are not directly apparent in the image of a handwritten word, as already suggested by Morasso et al. (1983). Strokes are partially hidden in the trajectory as a consequence of the superimposition process. To recover them, the pentip velocity has to be analyzed using an algorithm similar to the one described above.

Figure 3 clearly illustrates the concept of partially hidden strokes. The word 'she' as written by a subject on a digitizer (Fig. 3a) has been analyzed with the previous algorithm. Figures 3c and d show the original and reconstructed curvilinear and angular velocity signals respectively. What is of interest here is the representation of a possible action plan that could have been used by the subject to produce this word. The spatial and timing properties of the stroke sequence are illustrated in Fig. 3b and e respectively. To generate a continuous trace, 11 strokes have been used here. The subject started at the top of the letter 's', with the first stroke aimed at a first virtual target. Once this stroke is initiated, the subject assumes that this target distance will be covered with a certain degree of precision and after a certain movement time, as predicted by (5). The subject can thus anticipate the end of the first stroke and initiate the second stroke to aim at a second virtual target, and so on.

In this context, Fig. 3b can be seen as describing the spatial characteristics $(D_1, D_2, C_{0i}, \theta_{0i})$ of the action plan used by the subject, and Fig. 3e as describing the timing characteristic (t_{0i}) of the same plan, once activated. Moreover, the individual description of each stroke, in terms of μ_{ji} and σ_{ji}, allows a detailed analysis of the temporal properties of the neuromuscular agonist and antagonist systems used to execute that action plan. For example, for right-handers, strokes executed at 45° and 270° with respect to a baseline defined by the wrist oscillation plane are generally faster than strokes produced in other directions (Plamondon and Clément 1991). It is expected that these strokes will probably be characterized by smaller μ_j and σ_j values.

4 Psychophysical properties

One interesting feature of the handwriting model described in this study is that it is based on a kinematic theory of rapid human movements (Plamondon 1993c,d, 1995b,c, 1996, 1997a) that is consistent with the basic psychophysical observations that have been reported in the field. These observations follow from the delta-lognormal law which describes the magnitude of the velocity of each stroke. This results in an analytical description of a single stroke that can be used for the automatic analysis of complex trajectories. It is also useful for demonstrating how other well-known properties of handwriting simply emerge from the vectorial stroke summation.

4.1 Temporal and spatial control and invariance

According to the kinematic theory, the rules of stroke superimposition are governed by the basic speed/accuracy tradeoffs that result from the perceptivo-motor conditions (Plamondon 1993d, 1995c) that must be fulfilled to produce and control each individual stroke, as previously mentioned in Sect. 2, equation (5). In this perspective, the vectorial delta-lognormal model highlights the interactive role of shape, size and time in handwriting control. Many papers have been published on spatial or temporal invariance across changes in writing conditions (Freeman 1914; Denier van der Gon and Thuring 1965; Yasuhara 1975; Wing 1980; Hollerbach 1981; Stelmach and Teulings 1983; Greer and Green 1983; Thomassen and Teulings 1985; Wright 1990). Typical studies on these topics deal with specific measurements of spatiotemporal variables for words with a similar shape but of different size, words of similar size but a different shape, as well as words of similar shape written at different speeds. Some experimental conditions are also extended to study the effect of a change of effectors and of visual feedback (Bernstein 1967; Merton 1972; Wright 1990). The term 'word' is used here in a very broad sense and, depending on the study, refers to any piece of handwriting: a few strokes, a single letter, some alternating patterns, a sequence of letters or a few real and meaningful words or sentences.

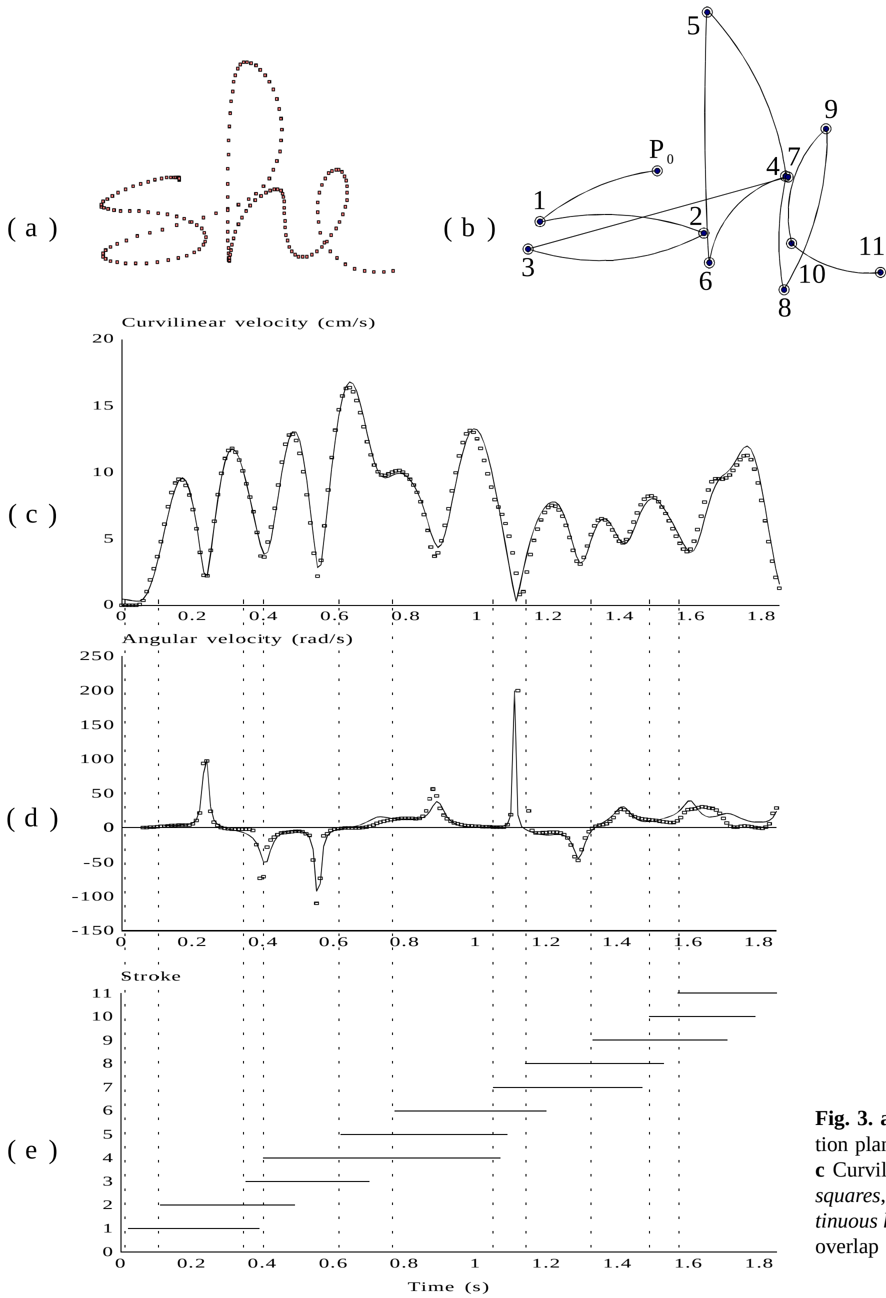

Fig. 3. a Original handwriting specimen (word ‘she’). **b** Action plan representation: strokes and virtual targets extracted. **c** Curvilinear velocity (cm/s): *continuous line*, original data; *squares*, model predictions. **d** Angular velocity (rad/s): *continuous line*, original data; *squares*, model predictions. **e** Time overlap of the stroke sequence

So far, some studies have supported the concept of motor equivalence, that is, the ability of a subject to write a word with a different group of end-effectors while roughly preserving its shape. Other studies show clearly that, for the same set of effectors, a subject is able to write the same word in different sizes while preserving its shape almost perfectly. With regard to time control, no consensus has been reached concerning what has been referred to as temporal invariance, that is, invariance of writing time across changes in writing size (Wright 1993).

The vectorial delta-lognormal model presented in this paper provides some clues to explain and interpret the various observations reported in these experiments. First, one must take into account a technical problem related to the measurements that are performed on the data used in these studies. According to the theory presented in this paper and in line with other similar studies (Morasso and Mussa Ivaldi 1982; Morasso et al. 1983; Morasso 1986), strokes are partially hidden in the signal and their individual shapes and timing properties are partially affected by the superimposition process. This basic phenomenon is generally neglected in the studies mentioned above, and operational methods to segment a handwritten trace are used. For example, a word is segmented at the minimum of its velocity pattern (e.g. Schomaker and Teulings 1991), at the minima of the Y coordinates (e.g. Mermelstein and Eden 1964) or at the points of peak curvature (e.g. Morasso 1986). These methods constitute a rapid and operational way to segment a trace, but they do not allow the recovery of the individual strokes that have been superimposed to generate the observed pattern. Mea-

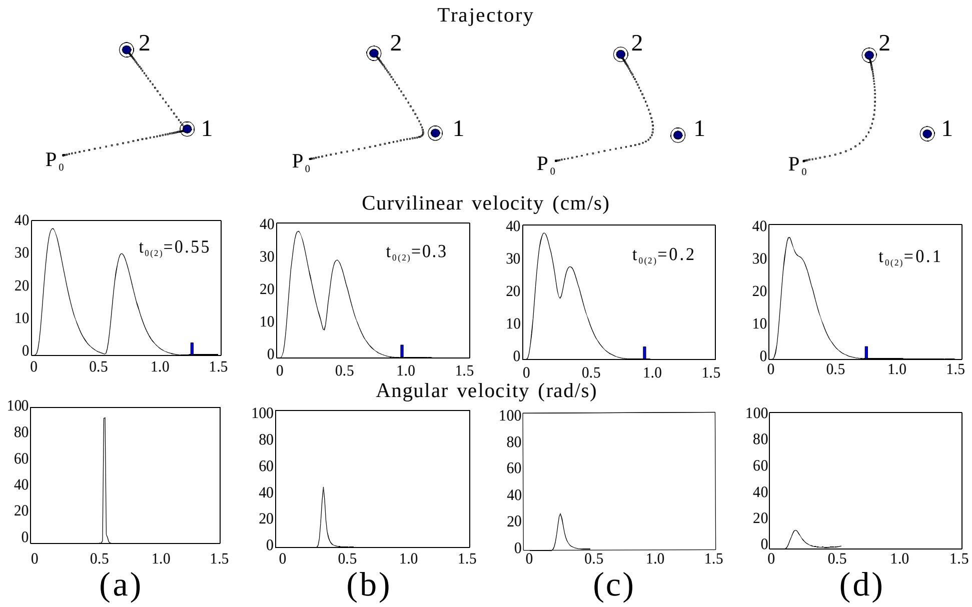

Fig. 4. a Sequence of two linear strokes with no superimposition: angular discontinuity, intermediate target reached. **b** Effect of a moderate stroke superimposition on the trajectory, the curvilinear and the angular velocities. **c** Effect of a significant stroke superimposition on the trajectory, the curvilinear and the angular velocities. **d** Effect of an almost complete superimposition of strokes on the trajectory, the curvilinear and the angular velocities

suring the time elapsed between two, three or a few of these operational segmentation marks, or evaluating the changes in the shape of the corresponding trace, reflects more or less what has happened at the stroke level, depending on the relative importance of the stroke superimposition process that has occurred in the production of a specific trace.

Looking at the simple traces described in Fig. 4, for example, it can be seen that, using the same pair of straight strokes, a different curved pattern can be generated depending on the time occurrence t_{02} of the second stroke (Guerfali 1996). If the resulting trace is segmented into 'operational strokes' using curvilinear velocity minima, different pairs of these strokes will be analyzed in terms of duration, size or shape, although the only effective change that occurred in this simulated example was to the parameter t_{02}. A single operational stroke might even be found when the superimposition process is very significant (Fig. 4d). If the same traces were analyzed in the representation space provided by the vectorial delta-lognormal model, using the segmentation method described in the previous section, a single pair of strokes would be theoretically extracted and analyzed in any of these specific conditions. Generally speaking, applying the kinematic theory to study spatio-temporal phenomena and comparing stroke timing, size and shape could provide better insight into the invariant properties of a trace and would reflect more clearly the possible action plan and strategies used by the subjects under different experimental conditions.

In this context, one can easily understand, for example, why some past studies have led to contradictory results regarding temporal invariance (Wright 1993). If we recall that each individual stroke is subject to a speed/accuracy trade-off (5), due to the delta-lognormal law that governs its velocity profile, then there is a relationship between stroke duration and the relative spatial accuracy of the target to be reached with that stroke (Plamondon 1993d, 1995c). The factor and exponent in this relationship (5) are related to the global timing properties of the agonist and antagonist neuromuscular networks involved in the production of the stroke. Temporal invariance requires at least the preservation of the relative spatial accuracy of the movement, at the stroke level, as well as keeping the neuromuscular parameters almost constant from trial to trial. Depending on the performance of the subjects with respect to these constraints, the expected temporal invariance will be more or less apparent.

We have shown in Figs. 2 and 3, for example, as well as in Guerfali (1996), Leclerc (1996), Plamondon and Guerfali (1996b), Plamondon et al. (1997) and Plamondon (1997b), comparative and quantitative results demonstrating the capacity of the vectorial delta-lognormal model to reconstruct real and complex data. The next step will be to use the vectorial delta-lognormal model to study large sets of real data to analyze and characterize subject performance in specific experiments (aimed at improving our understanding of the motor control aspects of handwriting), although this is beyond the scope of this paper. However, as in Hollerbach (1981), Morasso and Mussa-Ivaldi (1982), Edelman and Flash (1987), Bullock et al. (1993) and Singer and Tishby (1994), computer simulations can be used to highlight control strategies, supported by the vectorial delta-lognormal model, that are consistent with well-known behavioral data.

Figure 5 illustrates a few strategies that can be used to generate slightly different traces from the same action plan, that is, the same set of virtual targets, with a slightly different subset of strokes. Figure 5a shows a typical action plan made up of a sequence of 10 discontinuous strokes linking a set of 10 virtual targets. Figure 5b shows a typical continuous output, as obtained by activating a neuromuscular synergy with a proper timing scheme $(t_{0i}, \mu_{1i}, \mu_{2i}, \sigma_{1i}, \sigma_{2i})$. The stroke superimposition process results in a continuous trace: the word '*axe*'. The global writing time is given in the top right-hand corner. The same word (Fig. 5c) can be written in the same movement time but enlarged spatially by merely increasing the values of D_{1i} and D_{2i} of the action plan by a specific ratio (100% in this example). For many years, this phenomenon has been known as the isochrony principle and has been studied by many researchers (Binet and

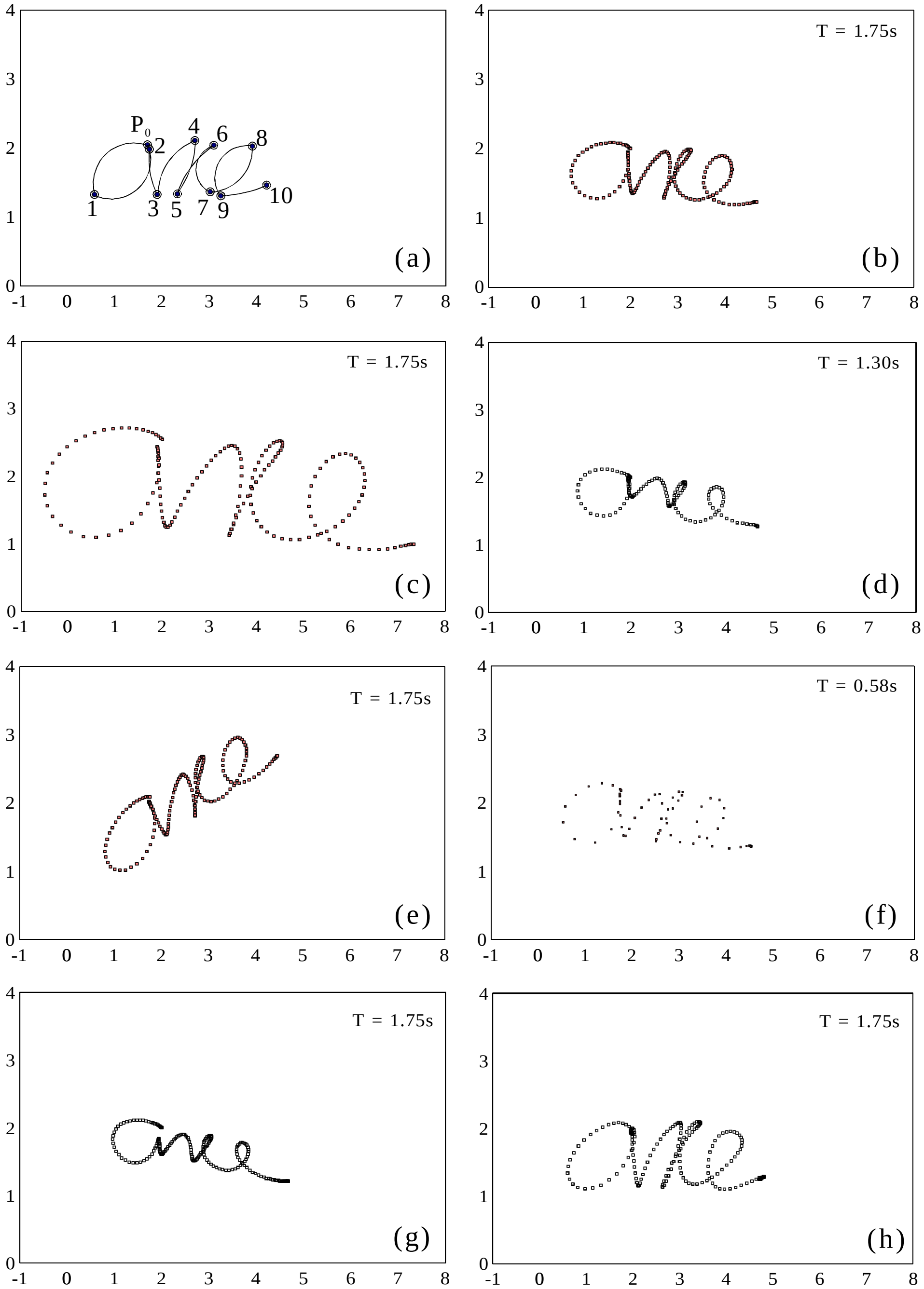

Fig. 5. a Action plan and virtual targets for the word 'axe' (10 strokes). **b** Typical output generated with the vectorial delta-lognormal model (D_{1i}/D_{2i} = 5.0). **c** Homothetic transform: 100% increase as obtained by increasing D_{1i} and D_{2i} by 100%). **d** Effect of decreasing relative spatial precision (D_{1i}/D_{2i} = 2.0) and stroke activation time t_{0i}. **e** Baseline change as obtained by increasing the θ_{0i} by 0.5 rad. **f** Increasing handwriting speed (by a factor of $\approx$ 300%), by decreasing μ_{1i}, μ_{2i} by ln (3) and t_{0i}, by 66%. **g** Effect of a simulated change of effectors: decreasing μ_{1i}, μ_{2i}, σ_{1i}, σ_{2i} by 20%. **h** Effect of a simulated change of effectors: increasing μ_{1i}, μ_{2i}, σ_{1i}, σ_{2i} by 20%

Courtier 1893; Freeman 1914; Fitts 1954; Viviani and McCollum 1983; Viviani 1986; Viviani and Schneider 1991). In this example, since the ratio D_{1i}/D_{2i} is kept constant, as are the other system parameters, the same relative spatial precision is reached for each stroke and the total movement time is constant. The movement time of the various operational strokes, as estimated, for example, using minima of curvilinear velocity, is also almost constant. This phenomenon was reported by Denier van der Gon and Thuring in 1965. In a similar fashion, in Fig. 5d, D_{1i} and D_{2i} are changed while keeping the individual stroke length ($D_{1i} - D_{12}$) constant, decreasing the ratio D_{1i}/D_{2i} and reducing the t_{0i}. The individual stroke duration is reduced and a loss in relative spatial precision will be observed, and the shape (and eventually the legibility) of the word will be affected.

A change in the baseline can be performed by modifying the θ_{0i} of the motor plan by a specific offset, as shown in Fig. 5e. Figure 5f illustrates another way to change writing speed using a mixed strategy. By reducing the t_{0i} as well as the μ_{1i}, μ_{2i}, a stiffer neuromuscular synergy that executes the sequence of strokes faster can be simulated. Finally, using the same action plan as in Fig. 5b but executing it with different neuromuscular synergies, one can mimic the effect of a change of effectors. In Fig. 5g and h, the μ_{1i}, μ_{2i}, σ_{1i} and σ_{2i} have been respectively decreased or increased by 20% as compared with the values used in Fig. 5b. The effector independence phenomenon as reported by Bernstein (1967) and

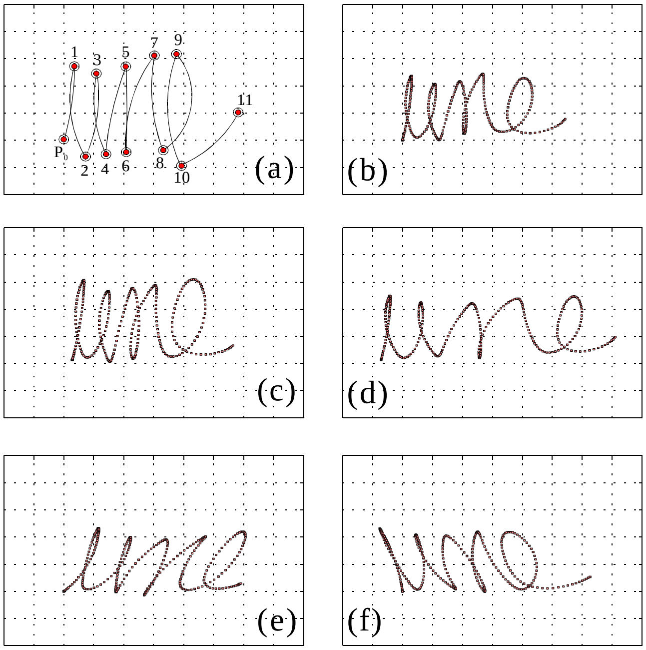

Fig. 6. a Action plan and virtual targets for the word 'une' (11 strokes). **b** Typical output generated with the vectorial delta-lognormal model. **c** Effect of a vertical increase in the virtual target positions. **d** Effect of a horizontal increase in the virtual target positions. **e** Slant change as obtained by a right translation of the top set of virtual targets. **f** Slant change as obtained by a left translation of the top set of virtual targets

Merton (1972) is clearly apparent here. The word 'axe' is still easy to read, although it appears slightly different from the specimen depicted in Fig. 5b. A larger modification of the timing properties of the agonist and antagonist systems would eventually result in a loss of legibility.

The vectorial delta-lognormal model also supports other well-known types of phenomena. Figure 6a shows another action plan, and Fig. 6b a typical instance of this plan. Figures 6c and d show the effect of increasing, respectively, the vertical and horizontal distances between the virtual targets of the action plan. Figures 6e and f depict the capacity of the model to change the writing slope by translating the top set of virtual targets to the right or to the left with respect to the bottom set of targets. In all these examples, the resulting traces look quite realistic. However, these vertical or horizontal scale changes are not as straightforward as the homothetic changes depicted in Fig. 5c. To simulate a change in the vertical scale, for example, one has to change more parameters of the action plan $(D_{1i}, D_{2i}, C_{0i}, \theta_{0i})$ as compared with the simple changes of D_{1i} and D_{2i} that were required to realize a full homothetic scaling. This observation is consistent with the fact that it is normally easier for human subjects to produce a full homothetic scaling as compared with a vertical or a horizontal change of scale or a change of slope. It must also be remembered that the problem is probably even more complex than the simple examples shown here, where the neuromuscular timing parameters $(\mu_{1i}, \mu_{2i}, \sigma_{1i}, \sigma_{2i})$ have been kept constant for simplicity.

As can be seen from these various simulated examples, different types of realistic patterns can be generated by the vectorial delta-lognormal model. In practice, a subject might use and combine some of these strategies to produce fluent and legible handwriting. Using the kinematic theory, the individual strokes could be recovered and their specific parameters analyzed under different experimental conditions to study these strategies.

4.2 Discontinuous angular control and the 2/3 power law

Another consequence of the vectorial delta-lognormal model is that the control of the angular velocity is not an independent process, as we have suggested in some of our previous studies (Plamondon 1987, 1989a, 1991, 1992). The continuous angular velocity signal emerges from the vectorial addition of discontinuous strokes. Depending on the timing difference between two successive strokes, different trajectories can be generated, with specific curvilinear and angular velocity patterns (Fig. 4). According to this view, a subject can generate different shapes with some characteristic velocity profiles, mostly by controlling the time occurrence t_{02} of the second stroke. The angular velocity of the pentip simply emerges from the stroke superimposition process, each individual stroke direction evolving according to the time integral described by (3).

This superimposition process provides an explanation for an important phenomenon that has been reported concerning the relationship between angular velocity and the curvature of a trajectory (or the curvilinear velocity and the radius of curvature) (for a review see Bourdon and Plamondon 1993). Indeed, Laquaniti et al. (1983, 1984) have observed that, for a certain class of movements (mostly elliptical or piece-wise elliptical trajectories), the curvature $C(t)$ and the angular velocity $(V_\theta(t))$ are linked by a 2/3 power law:

$$\nu_\theta(t) = kC^{2/3}(t) \tag{19}$$

or, if one analyzes the curvilinear velocity with respect to the radius of curvature of the trajectory:

$$\nu(t) = kR^{2/3}(t) \tag{20}$$

For more complex trajectories, this simple relationship becomes less valid and a better equation must be used to fit real data (Viviani and Schneider 1991; Viviani and Stucchi 1992):

$$\nu(t) = k(t)\left(\frac{R(t)}{1 - aR(t)}\right)^b \quad \text{or} \quad \nu_\theta(t) = k(t)\left(\frac{1}{C(t) + a}\right)^b \tag{21}$$

If $a = 0$ and $k(t)$ is a constant as a function of time, these latter relationships are equivalent to (19) and (20), for $b = 1/3$.

This 2/3 power law can be interpreted in the context of the vectorial delta-lognormal model (see Appendix). Indeed, the superimposition of two circular strokes leads to a complex trajectory that can be described by (12) and (16). A part of this trajectory approximates an ellipse and in this specific zone the 2/3 power relationship is valid. Outside the zone, the relationship will not be verified, a point that has been reported by a few studies (e.g. Thomassen and Teulings 1985; Wann et al. 1988).

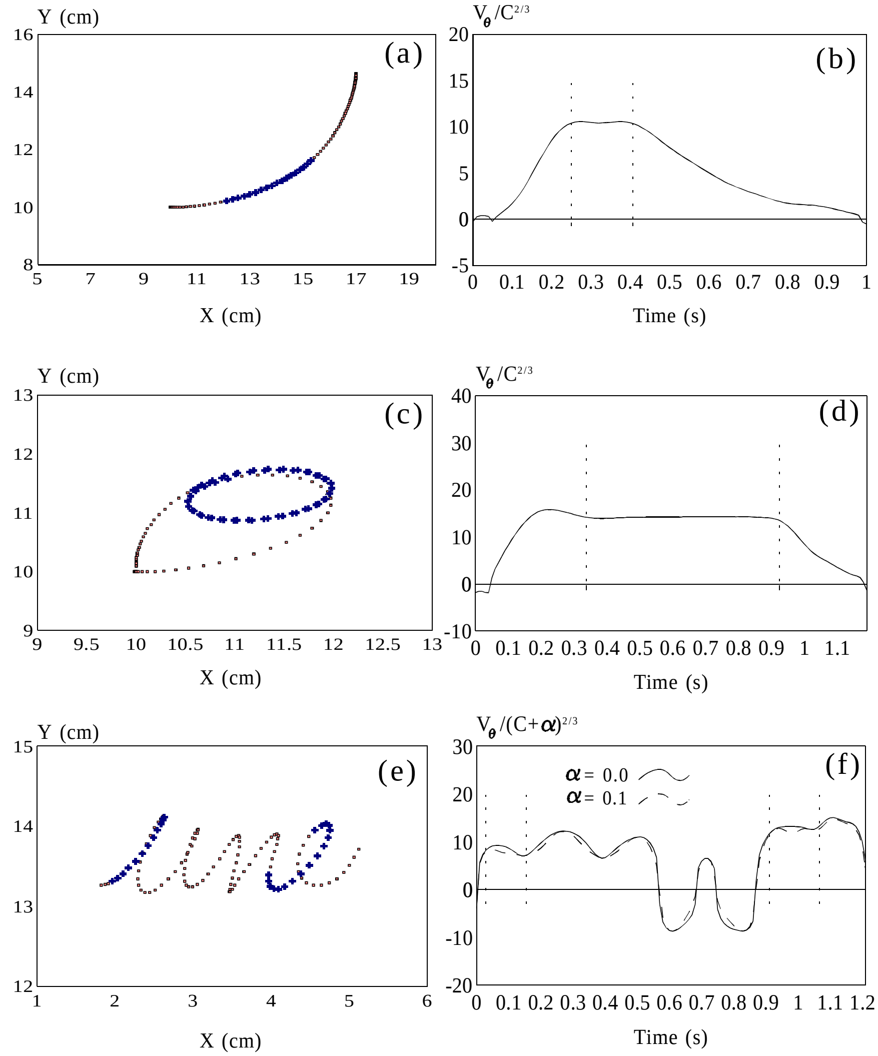

Fig. 7. a Superimposition of two strokes: *crosses*, portion of the simulated trajectory that respects the 2/3 power law. **b** Ratio $V_\theta/C^{2/3}$ as a function of time *continuous line*. *Vertical dotted lines*, limits of the duration when 2/3 power law applies. **c** Superimposition of six strokes: *crosses*, portion of the simulated trajectory that respects the 2/3 power law. **d** Ratio $V_\theta/C^{2/3}$ as a function of time *continuous line*. *Vertical dotted lines*, limits of the duration when the 2/3 power law applies. **e** Typical handwriting as produced by a human subject (word 'une'), and portions of the trajectory that best fits the 2/3 power law. **f** Ratio $V_\theta/(C + A)^{2/3}$ as a criterion of time *continuous line*. The portions of the trajectory that best fit the 2/3 power law (word 'une') are within the *vertical dotted lines*

Figure 7a shows a typical simulated trace, made up of the vectorial superimposition of two circular delta-lognormal strokes, and Fig. 7b depicts the ratio $\nu_\theta(t)/C(t)^{2/3}$ as a function of time. As can be seen, there is a specific period (identified by truncated vertical lines in Fig. 7b) during the execution of the movement where this ratio is almost constant and the corresponding portion of the trajectory (identified with large crosses in Fig. 7a) is roughly elliptical. Figures 7c and d illustrate how this phenomenon can become more apparent by repeating the same pattern of strokes while superimposing them to create a sequence of loops. For a large part of this latter trajectory, the 2/3 power law applies. By contrast, if one looks at the word 'une' as depicted in Fig. 7e, the 2/3 power law is quite a rough approximation, for some very specific portion of the trajectory. Even the more general equation (21) is only valid for very specific short moments (Fig. 7f). A more complex trajectory cannot be easily approximated by portions of ellipses, and the relationship between the curvilinear velocity and the radius of curvature is more general:

$$\nu(t) = f(R(t)) \tag{22}$$

This latter equation can be rewritten in the form of (21), which constitutes a good approximation for some specific portions of the trajectory.

5 Discussion

The vectorial delta-lognormal model can be used to represent and reconstruct handwriting both in the spatial and in the velocity domain. The model allows for the recovery of each individual stroke and its description with a specific set of parameters. It is then possible to study how these strokes are superimposed to produce a specific trace (Fig. 3). As such, the model provides an original and new set of analytical tools to analyze human movements (Plamondon 1997b), based on the realistic stroke description resulting from the kinematic theory of human movements (Plamondon 1993c,d, 1995b,c, 1996, 1997a).

Using analysis-by-synthesis methods to analyze neurobiological or neuropsychological data might provide new support for the theory, as well as limiting its range of application. From many points of view, the model is consistent with behavioral data, but cleverly designed experiments that exploit the model's global perspective might provide new insights into our understanding of motor control. The global approach, polarized around opposing agonist and antagonist systems, is in accordance with the recent comments made by Mackay (1997) and Sherwood (1997). The concept of planning movement in kinematic coordinates is also supported by some experiments (e.g. Wolpert et al. 1995), as is the idea that velocity might be the primary control variable in movement control (Flanders and Herman 1992; Soechting et al. 1995; Houk and Gibson 1987). But how does this happen? What is the role of the various motor cortex areas of the cerebellum, the basal ganglia, and the different motor neurons and muscle fibers? This has to be studied, and the use of the delta-lognormal law will be of great help for this purpose since it allows very good data fitting in three complementary representation spaces: curvilinear and angular velocity, and the static trajectory image.

A single stroke velocity profile, as described by a delta-lognormal law, is intrinsically asymmetric as compared, for example, with the symmetrical velocity profiles of the minimum snap (Edelman and Flash 1987) or minimum jerk models (Flash and Hogan 1985), the models that use cubic splines (Morasso and Mussa-Ivaldi 1982) or sinusoidal functions (Eden 1962; Hollerbach 1981). Moreover, the delta-lognormal velocity profiles are intrinsically continuous. Several previous models were based on discontinuous velocity profiles (Denier van der Gon et al. 1962; MacDonald 1964; Mermelstein and Eden 1964; Dooijes 1983; Flash and Hogan 1985; Plamondon and Lamarche 1986; Edelman and Flash 1987; Plamondon 1989a,b; Leclerc et al. 1992, Wada and Kawato 1995). Others use a continuous oscillatory pattern as a basic movement and generates discontinuities in the reconstructed signals (Eden 1962; Hollerbach 1980; Stettiner and Chazan 1994; Singer and Tishby 1994), discontinuities that are not apparent in the real signal.

Many models require advance definition of movement duration as a command parameter (Denier van der Gon et al. 1962; MacDonald 1964; Yasuhara 1975; Dooijes 1983; Flash and Hogan 1985; Plamondon and Lamarche 1986; Edelman and Flash 1987; Plamondon 1989a,b; Leclerc et al. 1992; Wada and Kawato 1995) and, as such, require a further hypothesis to deal with speed/accuracy trade-offs (Plamondon and Alimi 1997). With the kinematic theory, movement time does not need to be preplanned since it emerges from the delta-lognormal law that describes the stroke speed, predicts any forms of these trade-offs and suggests how they can constitute a key feature for the anticipation process that leads to stroke superimposition (Plamondon 1997b). According to this view, movement duration is not an explicit parameter of the motor action plan but an implicit characteristic of the trajectory. It emerges from the intrinsic properties of the delta-lognormal law characterizing a neuromuscular synergy. Consequently, in the time domain, fluent handwriting can be produced by voluntarily controlling the individual stroke starting time. The only other model that deals with the speed/accuracy trade-offs is the Bullock et al. (1993) model, provided that a proper set of GO signals is used as input and synchronized to control the three degrees of freedom required by this model.

The vectorial delta-lognormal model exploits the speed/accuracy trade-offs to define and represent a generic action plan in terms of a discontinuous sequence of circular strokes linking virtual targets that have to be reached within some spatial errors. This representation is consistent with several papers suggesting that arm movement planning occurs in a spatial coordinate system instead of a joint planning space (Morasso 1981; Abend et al. 1982). In practice, most of the virtual targets are not reached during execution due to the superimposition process which results in a continuous smooth trajectory. As a consequence, most of the virtual targets lie outside the real trajectory, and to recover them one has to look at high-curvature area and trajectory discontinuities (Leclerc 1996; Li et al. 1996). This is also consistent with the fact that human subjects tend to direct their eye saccades toward the same high curvature area when they look at a line image (Noton and Stark 1971). The sequence of directional targets required by the neural network model of Bullock et al. (1993) or the via points necessary to exploit the models based on minimization principles (Flash and Hogan 1985; Edelman and Flash 1987; Wada and Kawato 1995) do not necessarily have this perceptivo-motor interpretation, and this often leads to an ad-hoc representation of a specific trajectory.

Other differences between the vectorial delta-lognormal model and other models already published can be found at the spatio-temporal level. Indeed, the vectorial delta-lognormal model can easily take into account the various spatio-temporal properties observed in handwriting without requiring continuous monitoring of the current position of the end-effector, as required by some models (Bullock et al. 1993). For example, size and shift invariance can easily be obtained by modifying only two parameters. Some models ‘cannot uniformly regenerate an expanded or reduced trajectory of the original trajectory in time and space’ (Wada et al. 1995, p. 19). The vectorial delta-lognormal model explains the spatio-temporal properties of various trajectories in terms of changes in the parameters of the motor action plan, or at a lower level in terms of changes in the parameters of the neuromuscular networks that execute that plan, without requiring any minimization principle. It is not claimed here that the minimization of some kinematic or dynamic factors is not important in trajectory generation. Our interpretation is that these factors are not part of the basic stroke definition. Strokes are defined with respect to the fundamental asymptotic properties of the impulse response of a neuromuscular system and, consequently, this definition is sufficient to reconstruct a trajectory. However, it is clear that an analysis-by-synthesis experiment might lead to numerous equivalent solutions and the use of kinematic or dynamic minimization criteria might be one way to reduce the search space and to select an optimal solution with respect to such a criterion (Guerfali 1996).

6 Conclusion

In this paper we have presented the principal characteristics of a vectorial delta-lognormal model of handwriting generation to highlight how this model can be useful in the study and analysis of handwritten patterns and in understanding them. We have specifically focused on stroke description, superimposition and extraction to analyze and synthesize handwriting and to explain some of the most best-known psychophysical phenomena reported in the field. What consistently emerges from that approach is the fact that human beings seem to exploit the very basic properties and limitations of their own neuromuscular synergies, as described by the delta-lognormal law (Plamondon 1993c, 1995b), to produce complex tasks in a quasi-automatic fashion. In this perspective, the vectorial model is very powerful at a descriptive level and, as such, provides a global framework and a general methodology for pointing out and exploiting among other things, the higher-level strategies used by some subjects to accomplish these tasks. Although the algorithms are complex and the solutions might not be unique (Guerfali 1996), being able to perform a reverse analysis and extract the original intended strokes from a specific trajectory is a key tool that can be very useful to researchers interested in studying fundamental aspects of handwriting control as well as in applying these concepts to the design of computer-based systems that use handwriting or gesture as a human-machine interface.

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Appendix. The origin of the 2/3 power law

The vectorial delta-lognormal model predicts under which conditions a 2/3 power law will be observed between the angular velocity and the trajectory curvature. This can be seen by analyzing a trajectory made up of two strokes superimposed in time. Each stroke 'i' is characterized by a curvature C_{0i} and an initial angular direction θ_{0i}, and the magnitude of its velocity vector is described by a $\Delta\Lambda$ law.

Assuming for simplicity that the first stroke starts at $t_{01} = 0$ and the second at t_{02}, the velocity profile of the complex profile is described by

$$\vec{\nu}(t) = \vec{\nu}_1(t) + \vec{\nu}_2(t - t_{02}) \tag{A1}$$

If this vector is analyzed in a cartesian coordinate system, the X and Y components of the resulting velocity are obtained by

$$\nu_x(t) = \nu_1(t)\cos(\theta_1(t)) + \nu_2(t - t_{02})\cos(\theta_2(t - t_{02})) \tag{A2}$$

$$\nu_y(t) = \nu_1(t)\sin(\theta_1(t)) + \nu_2(t - t_{02})\sin(\theta_2(t - t_{02})) \tag{A3}$$

If $\Delta\theta(t)$ is defined as the instantaneous difference in angular direction between the two strokes at time t:

$$\Delta\theta(t) = \theta_1(t) - \theta_2(t - t_{02}) \tag{A4}$$

(A2) can be written as:

$$\nu_x(t) = \nu_1(t)\cos(\theta_1(t)) - \nu_2(t - t_{02})\cos(\theta_1(t) - \Delta\theta(t)) \tag{A5}$$

and using trigonometric transformations

$$\begin{aligned}\nu_x(t) = {} & \nu_1(t)\cos(\theta_1(t)) + \nu_2(t - t_{02})\cos(\theta_1(t))\cos(\Delta\theta(t)) \\ & -\nu_2(t - t_{02})\sin(\theta_1(t))\sin(\Delta\theta(t))\end{aligned} \tag{A6}$$

Similarly, for the Y component, we obtain

$$\begin{aligned}\nu_y(t) = {} & \nu_1(t)\sin(\theta_1(t)) + \nu_2(t - t_{02})\sin(\theta_1(t))\cos(\Delta\theta(t)) \\ & -\nu_2(t - t_{02})\cos(\theta_1(t))\sin(\Delta\theta(t))\end{aligned} \tag{A7}$$

Hypothesis 1: There are parts of the resulting trajectory where

$$\Delta\theta(t) \simeq \frac{n\pi}{4} \tag{A8}$$

For that portion, (A6) and (A7) reduce to

$$\nu_x(t) \simeq \nu_1(t)\cos(\theta_1(t)) + \nu_2(t - t_{02})\sin(\theta_1(t)) \tag{A9}$$

$$\nu_y(t) \simeq \nu_1(t)\sin(\theta_1(t)) - \nu_2(t - t_{02})\cos(\theta_1(t)) \tag{A10}$$

where we focus on $n = 1$ for the rest of the demonstration.

Using trigonometric transformations, these latter two equations can be rewritten as follows:

$$\nu_x(t) \simeq \nu_1(t)\cos(\theta_1(t)) + \nu_2(t - t_{02})\cos(90^\circ - \theta_1(t)) \tag{A11}$$

$$\nu_y(t) \simeq \nu_1(t)\sin(\theta_1(t)) - \nu_2(t - t_{02})\sin(90^\circ - \theta_1(t)) \tag{A12}$$

Hypothesis 2: The starting time t_{02} of the second stroke is such that the following equivalences are roughly valid on the specific portion of the trajectory under study:

$$\nu_2(t - t_{02})\cos(90^\circ - \theta_1(t)) \simeq K_x(t)\nu_1(t)\cos(\theta_1(t)) \tag{A13}$$

$$\nu_2(t - t_{02})\sin(90^\circ - \theta_1(t)) \simeq K_y(t)\nu_1(t)\sin(\theta_1(t)) \tag{A14}$$

In other words, the subject starts his or her second stroke at a specific time, in such a way as to compensate for the 90° phase shift between the two strokes and to link the velocity module of this second stroke to the velocity module of the first stroke by a time function $K_x(t)$ or $K_y(t)$.

Thus, the X and Y components of the resulting velocity can be described by

$$\nu_x(t) \simeq \nu_1(t)(1 + K_x(t))\cos(\theta_1(t)) \tag{A15}$$

$$\nu_y(t) \simeq \nu_1(t)(1 + K_y(t))\sin(\theta_1(t)) \tag{A16}$$

The previous two equations have a form similar to that of the parametric equation of the velocity of a particle along an elliptic trajectory, provided that

$$\nu_1(t)(1 + K_x(t)) \simeq a \tag{A17}$$

$$\nu_1(t)(1 + K_y(t)) \simeq b \tag{A18}$$

where a and b are approximately constant for that portion of the trajectory. (The reader should note that if $a > 0$ and $b < 0$ the trajectory will be hyperbolic and the same conclusion will apply.)

If these latter two conditions (A17 and A18) are met, then the curvature $C(t)$ of that elliptical portion will be

$$C(t) = \frac{ab}{(\nu(t))^3} \tag{A19}$$

and

$$\nu(t) \simeq \left[\nu_1(t)^2(1 + K_x(t))(1 + K_y(t))\right]^{1/3} R(t)^{1/3} \tag{A20}$$

that is

$$\nu(t) \propto R^{1/3}(t) \tag{A21}$$

or

$$\nu_\theta(t) \propto C^{2/3}(t) \tag{A22}$$

Thus, under the specific conditions described by hypotheses 1 and 2, the vectorial delta-lognormal model predicts that the time superimposition of two circular strokes will result in a complex trajectory having a portion that can be approximated by an ellipse, and that, for that specific portion, a 2/3 power law will be observed between the angular velocity and the curvature. Outside that specific portion, the law will not be valid. Computer simulations show that the length of the trajectory where the 2/3 power law applies can be expanded if more than two strokes are superimposed (see Fig. 7d).

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