1. Improving Hollerbach's model

a. The Hollerbach model

One of the first, if not the first, oscillatory model of handwriting was proposed by Hollerbach [1981]. It comes jointly with a spring-mass model of the arm apparatus, which will not be presented here.

In this model, handwriting is seen as the result of two superimposed oscillators on two distinct directions of the plane. Although any non-sinusoidal oscillators could work as well, it is more convenient to use sinusoids. Moreover, this choice was more compliant with the spring muscle model. Oscillators time evolution is defined as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a\sin(\omega_x t + \phi_x) + c \qquad (1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = b\sin(\omega_y t + \phi_y) \qquad (2)$$

where a and b are the horizontal and vertical velocity amplitudes, ω_x, ω_y, ϕ_x and ϕ_y are respectively the frequencies and the phases associated to these directions. c represent the constant displacement to the right when writing. The directions on which oscillators move are not necessarily chosen. It would be advocable to choose the horizontal axis for one of them and the slant direction for the other

Model parameters (ie. a, b, ω_x, ω_y, ϕ_x and ϕ_y) are supposed to be piecewise constant. All these parameters change when the vertical velocity is null.

Interestingly, the slant is described by the angle β, which can be expressed as:

$$\tan\beta = \frac{b}{a\cos\phi} \text{where} \phi = \phi_x - \phi_y \qquad (3)$$

Another interesting value is the value of the horizontal velocity when the vertical velocity is null:

$$\Psi = \frac{\mathrm{d}x}{\mathrm{d}t}(t_{y_0}) = c - a\sin\phi \qquad (4)$$

The sign and magnitude of this value indicates the particular shape of the trace at this point. If Ψ is next to zero then the top corner will look sharp. If it is positive, the top corner will become rounded. Conversely, a negative value of Ψ will result in a full loop. This behavior is shown in table 1.

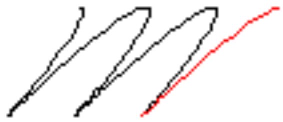
Shape	Ψ
	1.262
	-43.74
	37.41

Table 1: The shape of the top corners depends on the sign and magnitude of Ψ.

b. Improving the oscillatory model

Hollerbach's model does not address important questions:

- Is there a way to quickly extract parameters, given a recorded trace?
- Is the ad-hoc c parameter really necessary?
- Parameters (a, b, ω_x, ω_y, Φ_x and Φ_y) are only allowed to change at times of zero-crossing vertical velocity: is there any reason for this dissymmetry between x and y axes?

The main computational drawback of Hollerbach's model is the way we can get the parameters from the trace: it is a non-linear curve-fitting problem. Usual optimization methods are costly; we will present an algorithm that is much more efficient. This efficiency is possible because of our choice (that will be presented later) of the moments when parameters are allowed to change.

The c parameter is of course representing the constant drift of the hand from left to right when writing, but it seems a rather ad-hoc way for taking it into account. Moreover, it imposes the X-axis to be exactly aligned with the trace direction, and it introduces a dissymmetry between X and Y. We will show that it can be simply omitted, and that the drift can be accounted for by cumulating phase values between cycles.

Finally, the fact that changes on X and Y parameters occur simultaneously, when Y velocity is null, make the Hollerbach model a stroke-by-stroke approach: even though each stroke is formed via continuous harmonic movements on X and Y, it is simply concatenated to the next, which may be very different in shape, leading to undesired angularities at the junction point. This approach lacks *co-articulation*, that is to say continuity between the successive movements of writing. We will show how our improved model overcomes this problem.

2. The Improved Hollerbach model

Our improved version of Hollerbach's model is both simpler, symmetric and allows for a fast parameters extraction algorithm. It is completely described in the following paragraphs:

a. Cinematic equations

We use the canonical orthogonal directions of the plan space as axes for the two oscillators. The movements on the x and y axes are defined by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_x(\mathrm{t})\sin(\omega_x(t).t + \phi_x(\mathrm{t})) \tag{3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a_y(\mathrm{t})\sin(\omega_y(t).t + \phi_y(\mathrm{t})) \tag{4}$$

where $a_x, \omega_x, \phi_x, a_y, \omega_y, \phi_y$ are constant-by-part functions defined by a parameter series described in b-

b. Parameter series

Let $t_{x,0}, \dots, t_{x,N_x}$ be the moments of zero-velocity on the x-axis, and $t_{y,0}, \dots, t_{y,N_y}$ the moments of zero-velocity on the y-axis.

We compute the following series elements:

$$\omega_{x,i} = \frac{\pi}{t_{x,i+1}-t_{x,i}} \tag{5}$$

$$\phi_{x,i} = -\frac{\pi.t_{x,i}}{t_{x,i+1}-t_{x,i}} \tag{6}$$

$$a_{x,i} = sign(\frac{\mathrm{d}x}{\mathrm{d}t})_{[t_{x,i}-t_{x,i+1}[}.\, mean(\frac{\mathrm{d}x}{\mathrm{d}t})_{[t_{x,i}-t_{x,i+1}[}.\, std(\frac{\mathrm{d}x}{\mathrm{d}t})_{[t_{x,i}-t_{x,i+1}[} \tag{7}$$

$$\omega_{y,i} = \frac{\pi}{t_{y,i+1}-t_{y,i}} \tag{8}$$

$$\phi_{y,i} = -\frac{\pi.t_{y,i}}{t_{y,i+1}-t_{y,i}} \tag{9}$$

$$a_{y,i} = sign(\frac{\mathrm{d}y}{\mathrm{d}t})_{[t_{y,i}-t_{y,i+1}[}.\, mean(\frac{\mathrm{d}y}{\mathrm{d}t})_{[t_{y,i}-t_{y,i+1}[}.\, std(\frac{\mathrm{d}y}{\mathrm{d}t})_{[t_{y,i}-t_{y,i+1}[} \tag{10}$$

where $mean$ and std are respectively the mean and standard deviation values on an interval.

The functions $a_x, \omega_x, \phi_x, a_y, \omega_y, \phi_y$ are defined by:

$$a_x(t) = a_{x,i} \qquad t_{x,i} \le t \le t_{x,i+1} \tag{11}$$

$$\omega_x(t) = -\frac{\phi_{x,i}}{t_{x,i}} \qquad t_{x,i} \le t \le t_{x,i+1} \tag{12}$$

$$\phi_x(t) = \phi_{x,i} \qquad t_{x,i} \le t \le t_{x,i+1} \tag{13}$$

$$a_y(t) = a_{y,i} \qquad t_{y,i} \le t \le t_{y,i+1} \tag{14}$$

$$\omega_y(t) = -\frac{\phi_{y,i}}{t_{y,i}} \qquad t_{y,i} \le t \le t_{y,i+1} \tag{15}$$

$$\phi_y(t) = \phi_{y,i} \qquad t_{y,i} \le t \le t_{y,i+1} \tag{16}$$

Finally a trace is completely defined by the two series:

$$\left(t_{x,i}, a_{x,i}, \omega_{x,i}, \phi_{x,i}\right) \qquad 0 \le i \le N_x \tag{17}$$

$$\left(t_{y,i}, a_{y,i}, \omega_{y,i}, \phi_{y,i}\right) \qquad 0 \le i \le N_y \tag{18}$$

3. Justification

a. Computation of ω and ϕ

Between two successive zeros $t_{x,i}$ and $t_{x,i+1}$ of the velocity function $\frac{\mathrm{d}x}{\mathrm{d}t}$ of x performs half a period of the sinus function, so:

$$t_{x,i+1} - t_{x,i} = \frac{\pi}{\omega_{x,i}} \quad \text{and} \quad \omega_{x,i}.t_{x,i} + \phi_{x,i} = 0$$

Similarly on the Y-axis:

$$t_{y,i+1} - t_{y,i} = \frac{\pi}{\omega_{y,i}} \quad \text{and} \quad \omega_{y,i}.t_{y,i} + \phi_{y,i} = 0$$

Hence formulas (5), (6) and (8), (9).

b. Computation of $a_{x,i}$

Consider the following function:

$$f: x \mapsto a\sin(\omega x + \phi)$$

where a, ω and φ are independent to x. First, let us calculate the mean and variance of f between two successive zeros:

$$M = \frac{\pi}{\omega}\int_{\frac{-\phi}{\omega}}^{\frac{\pi-\phi}{\omega}} f(x)dx = \frac{2a}{\pi}$$

$$V = \frac{\pi}{\omega}\int_{\frac{-\phi}{\omega}}^{\frac{\pi-\phi}{\omega}} (f(x) - M)^2 dx = \frac{a^2(-8+\pi^2)}{2\pi^2}$$

Then, let us add the calculated mean and the square root of the calculated variance (ie. the standard deviation) and divide the result by a:

$$R = M + \sqrt{V} = 2\,\frac{a}{\pi} + \frac{\sqrt{2}\sqrt{a^2(-8+\pi^2)}}{2\pi}$$

$$\frac{R}{a} = \frac{4+\sqrt{2}\sqrt{-8+\pi^2}sgn(a)}{2\pi}$$

Which if we give a numerical approximation leads to (if a is positive):

$$\frac{R}{a} \approx 0.944$$

This result shows that the amplitude of a sinusoidal signal can be approximated by the sum and standard deviation of this signal on a semi-period (zero to zero), independently of the frequency and phase, hence formulas (7) and (10).

c. Minimum number of parameters

Following (Teulings, 1996), it is interesting to assess the minimum number of parameter updates necessary to reconstruct a particular handwriting sample.
The 2 parameter series (17) and (18) are redundant since they can be completely reconstructed from the following elements:

- The initial values $(t_{x,0}, a_{x,0}, \omega_{x,0}, \phi_{x,0})$ and $(t_{y,0}, a_{y,0}, \omega_{y,0}, \phi_{y,0})$
- The series $(a_{x,i}, \omega_{x,i}), 0 \le i \le N_x$ and $(a_{y,i}, \omega_{y,i}), 0 \le i \le N_y$

The model is then fairly parsimonious, as it needs only two values for every change.

4. Parameter extraction algorithm for a sampled trace

Suppose the recorded handwriting sample is represented by a chronological finite list of time stamped positions:

$$S = (t_i, x_i, y_i)_{0 \le i \le N, N \in \mathbb{N}^*, \forall i > 0, t_i > t_{i-1}}$$

We describe here an algorithm for extracting from this recorded trace the series of parameters

$$\left(t_{x,i}, a_{x,i}, \omega_{x,i}, \phi_{x,i}\right) \qquad 0 \le i \le N_x \qquad (17)$$

$$\left(t_{y,i}, a_{y,i}, \omega_{y,i}, \phi_{y,i}\right) \qquad 0 \le i \le N_y \qquad (18)$$

Step 1

$x = (x_i)_{0 \le i \le N}$ is differentiated according to $t = (t_i)_{0 \le i \le N}$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}}\right)_{0 < i \le N}$$

$y = (y_i)_{0 \le i \le N}$ is differentiated according to $t = (t_i)_{0 \le i \le N}$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right)_{0 < i \le N}$$

Step 2

Zeros are added to the beginning and to the end of the derivative signal. From a theoretical point of view this could be contested: it is clear (for example if you look at the pressure of the pen) that velocity is not always null when a writer begins or ends a trace; but we cannot infer this information from the time stamped positions only.

Step 3

A zero-crossing algorithm is applied on the derivative, which is low-pass filtered firstly (using a flat window of size 6). This prevents this algorithm to find clusters of zeros due to acquisition irregularities or noise. It yields the two series $t_{x,0}, \dots, t_{x,N_x}$ and $t_{y,0}, \dots, t_{y,N_y}$

Step 4

The parameter series elements are computed:

$$\omega_{x,i} = \frac{\pi}{t_{x,i+1} - t_{x,i}}, \quad 0 \le i \le N_x$$

$$\phi_{x,i} = -\frac{\pi . t_{x,i}}{t_{x,i+1} - t_{x,i}}, \quad 0 \le i \le N_x$$

$$a_{x,i} = sign\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{[t_{x,i} - t_{x,i+1}[} . \, mean\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{[t_{x,i} - t_{x,i+1}[} . \, std\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{[t_{x,i} - t_{x,i+1}[}, \; 0 \le i \le N_x$$

$$\omega_{y,i} = \frac{\pi}{t_{y,i+1} - t_{y,i}}, \quad 0 \le i \le N_y$$

$$\phi_{y,i} = -\frac{\pi . t_{y,i}}{t_{y,i+1} - t_{y,i}}, \qquad 0 \le i \le N_y$$

$$a_{y,i} = sign\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_{[t_{y,i} - t_{y,i+1}[} . \, mean\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_{[t_{y,i} - t_{y,i+1}[} . \, std\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_{[t_{y,i} - t_{y,i+1}[}, \; 0 \le i \le N_y$$

Step 5

The X and Y velocities are reconstructed, using equations (3) and (4) and given the computed parameter series. Then X and Y traces are reconstructed by integration of the velocities.